2015-2016 – Econ590
Topics in Macroeconomics

Simple Dynamics with Strategic Interactions

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Simple Dynamics with Strategic Interactions

Environment

- N firms
- Each firm accumulates a capital good in quantity X_i by investing I_i
- Decision rule and law of motion for X are

$$X_{it+1} = (1-\delta)X_{it} + I_{it} \tag{1}$$

$$I_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1} + F(I_t) \tag{2}$$

  with $I_t = \frac{\sum I_{jt}}{N}$
- all parameters are positive, $\delta < 1$, $\alpha_1 < 1$, $\alpha_2 < 1$.
- $\alpha_1 X_{it}$: Optimal size argument
- $\alpha_2 I_{it-1}$: Adjustment cost argument
- $F(I_t)$: Strategic Interactions
  - $F' > 0$: strategic complementarities
  - $F' < 0$: strategic substitutabilities

Simple Dynamics with Strategic Interactions

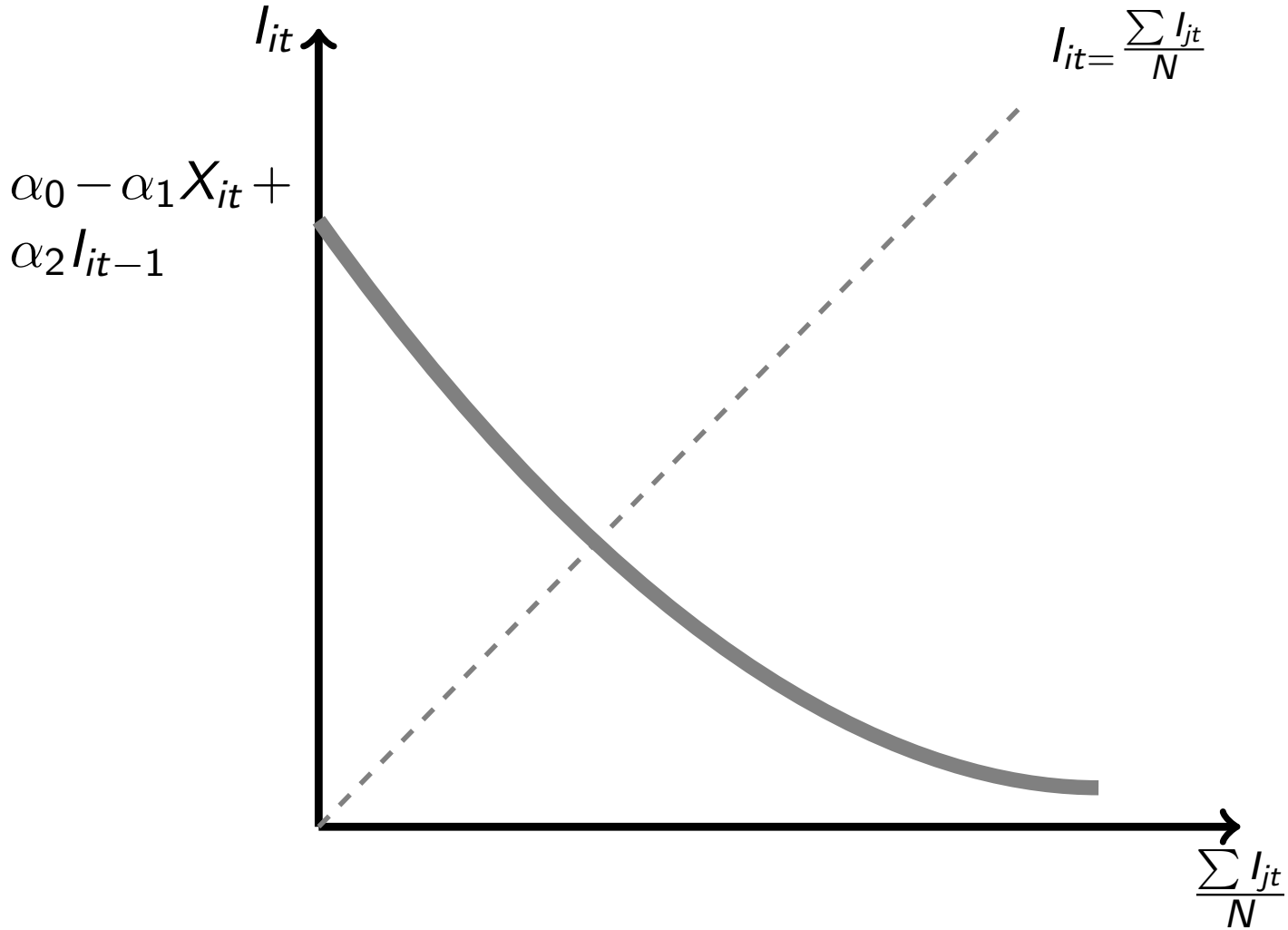
Environment

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + I_{it} \\ I_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1} + F(I_t) \end{aligned}$$

- Define (X^s, I^s) as the steady state of the linear model where $F(\cdot) = 0$
- Normalize $F(I^s) = 0$

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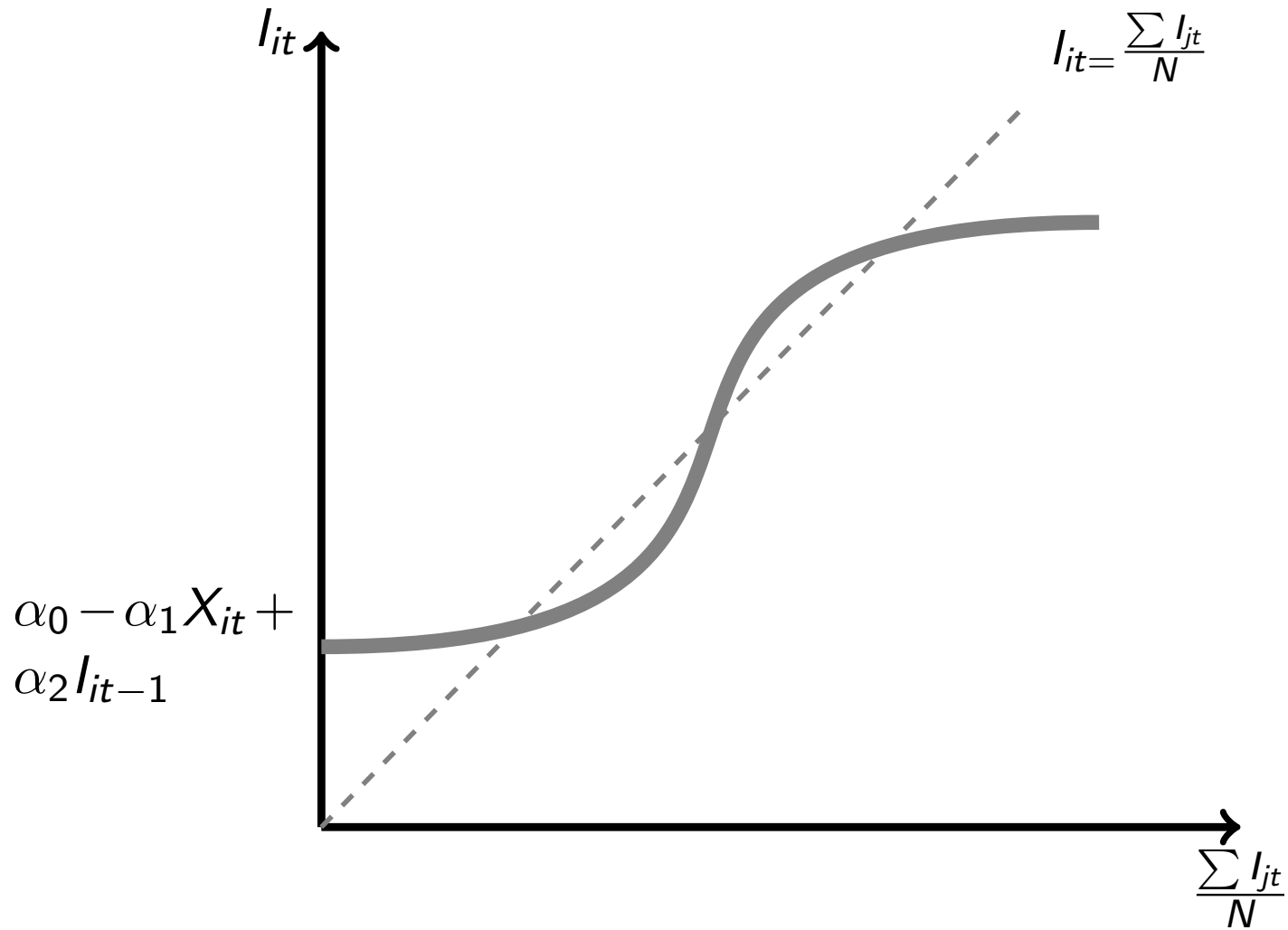
"Best response" rule for a given history - Strategic substitutability



$$I_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1} + F\left(\frac{\sum I_{jt}}{N}\right)$$

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"Best response" rule for a given history - Multiple Equilibria

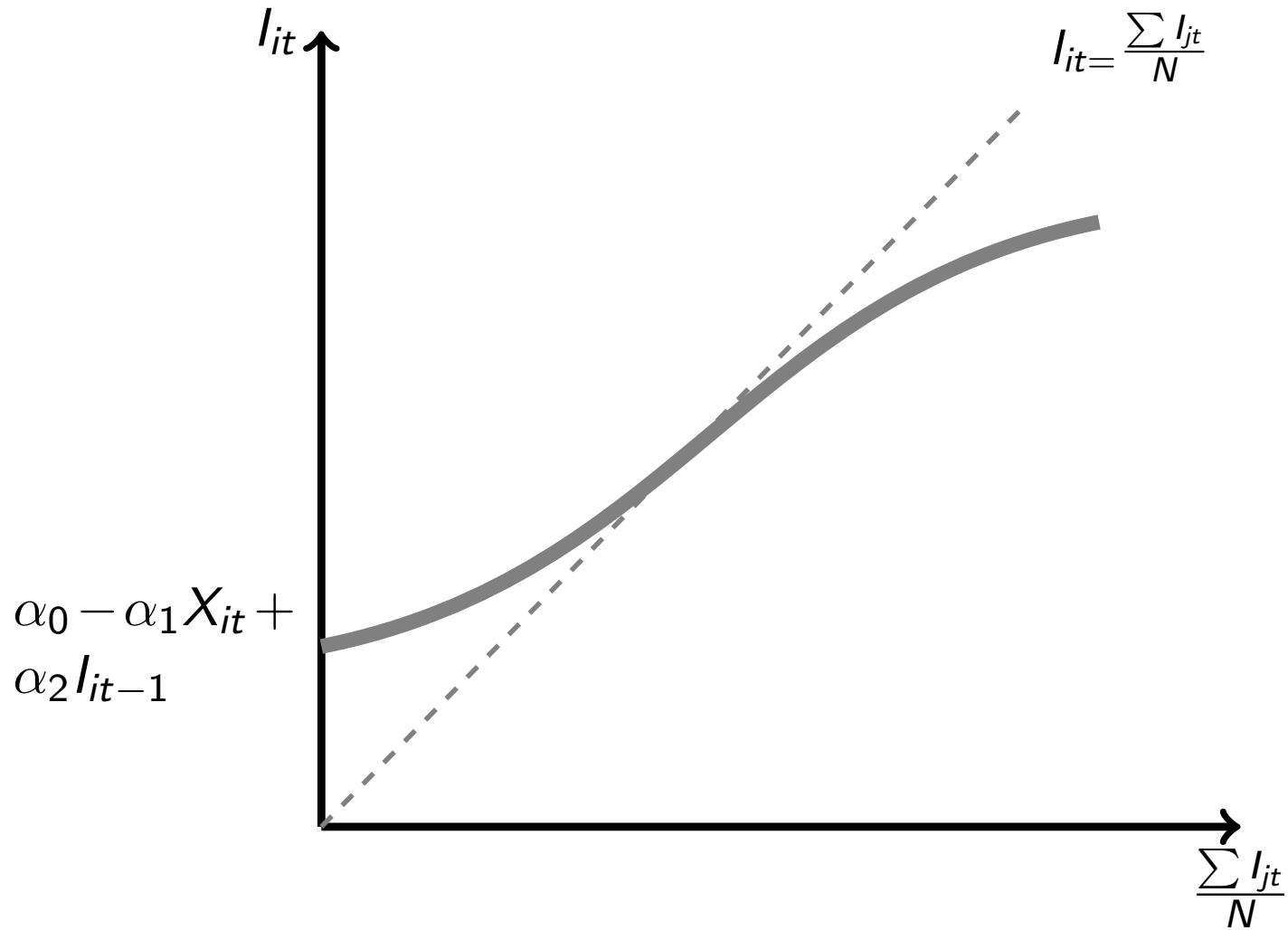


$$I_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1} + F\left(\frac{\sum I_{jt}}{N}\right)$$

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"Best response" rule for a given history - Strategic complementarity

- We restrict to the case where $F'(\cdot) < 1$, so that there are never multiple equlibria *within* period t



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One-Dimension Case

- Assume $\alpha_2 = 0$
- Then the model is

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + I_{it} \\ I_{it} &= \alpha_0 - \alpha_1 X_{it} + F(I_t) \end{aligned}$$

- It boils down to one order one dynamic equation in X_{it}

$$X_{it+1} = \alpha_0 + (1-\delta-\alpha_1)X_{it} + F(X_{t+1} - (1-\delta)X_t)$$

- So that, for symmetrical allocations

$$X_{t+1} = \alpha_0 + (1-\delta-\alpha_1)X_t + F(X_{t+1} - (1-\delta)X_t)$$

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One-Dimension Case

- Assume no strategic interactions : $F(\cdot) \equiv 0$

$$X_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)X_t$$

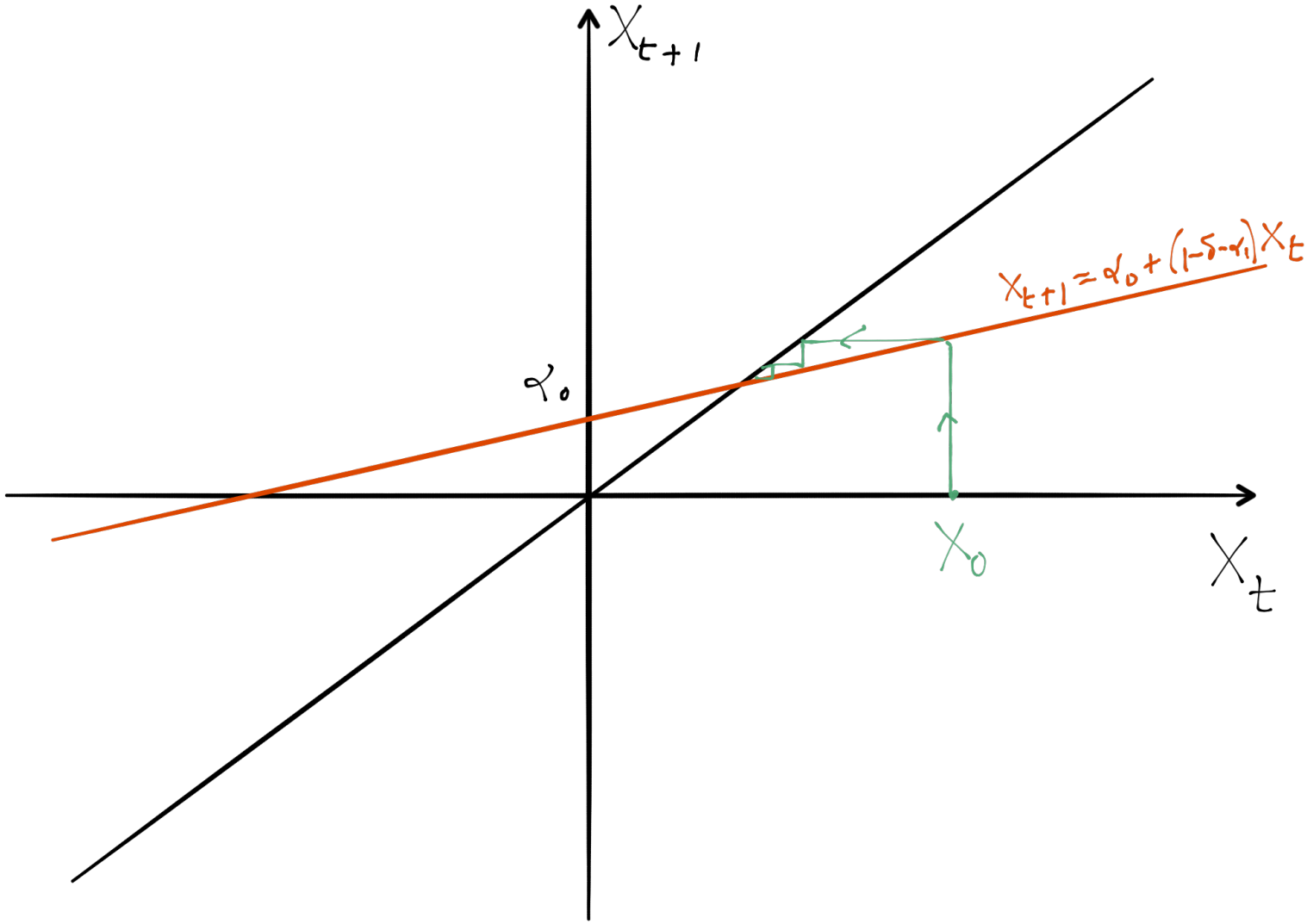
- Note that $(1 - \delta - \alpha_1)$ can be either > 0 or < 0.

Proposition 1

The Economy is globally stable as $|1 - \delta - \alpha_1| < 1$

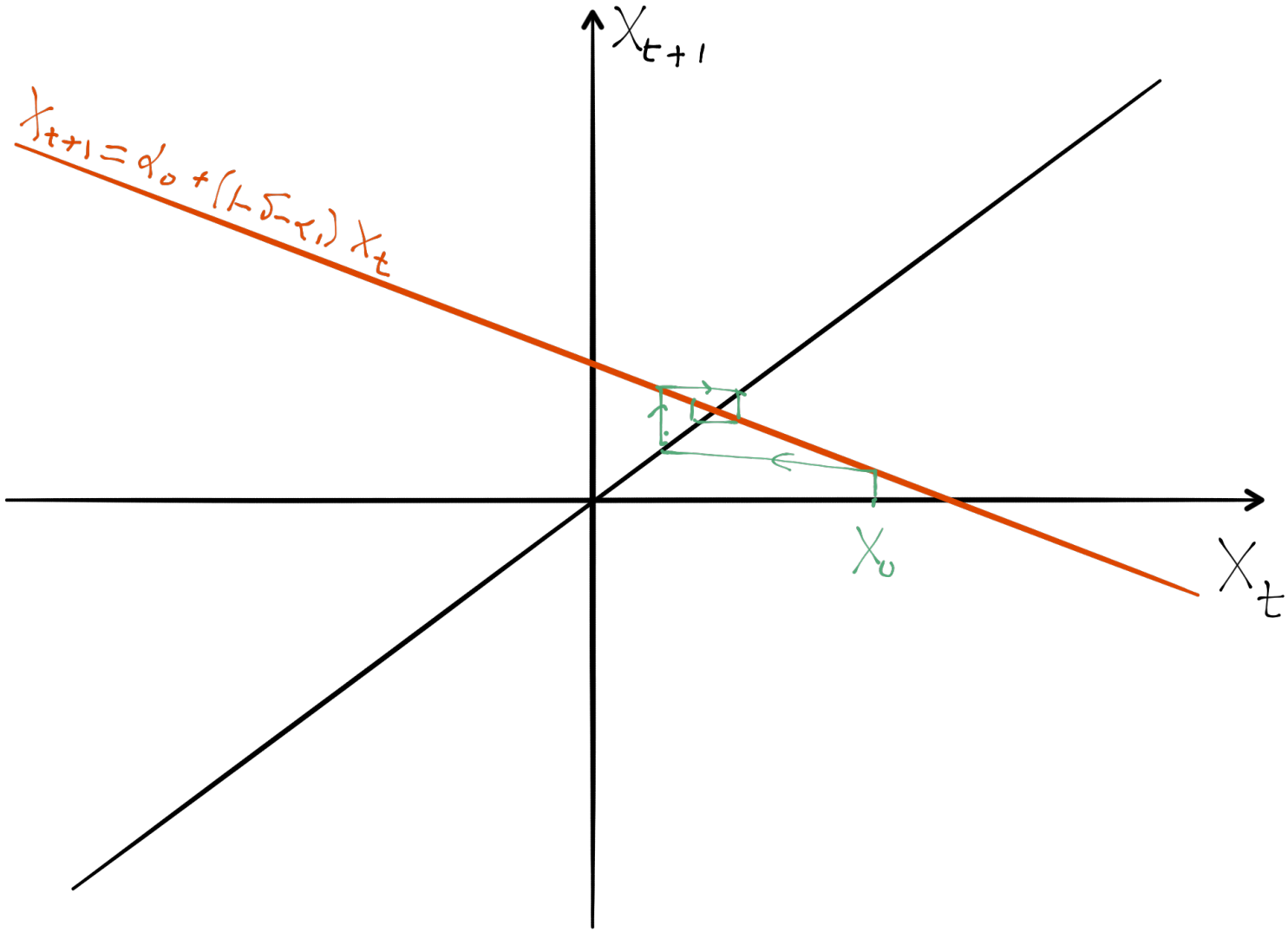
Simple Dynamics with Strategic Interactions

One-Dimension Case



Simple Dynamics with Strategic Interactions

One-Dimension Case



Simple Dynamics with Strategic Interactions

One-Dimension Case

- Reintroduce strategic interactions

$$X_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)X_t + F(X_{t+1} - (1 - \delta)X_t)$$

- Linearize around the steady state X^s

$$X_{t+1} = (\alpha_0 + \delta X^s) + (1 - \delta - \alpha_1)X_t + F'(X^s)(X_{t+1} - (1 - \delta)X_t)$$

- so that

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(X^s)} + \frac{((1 - \delta)(1 - F'(X^s)) - \alpha_1)}{1 - F'(X^s)} X_t$$

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One-Dimension Case

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(X^s)} + \frac{((1-\delta)(1-F'(X^s)) - \alpha_1)}{1 - F'(X^s)} X_t$$

Proposition 2

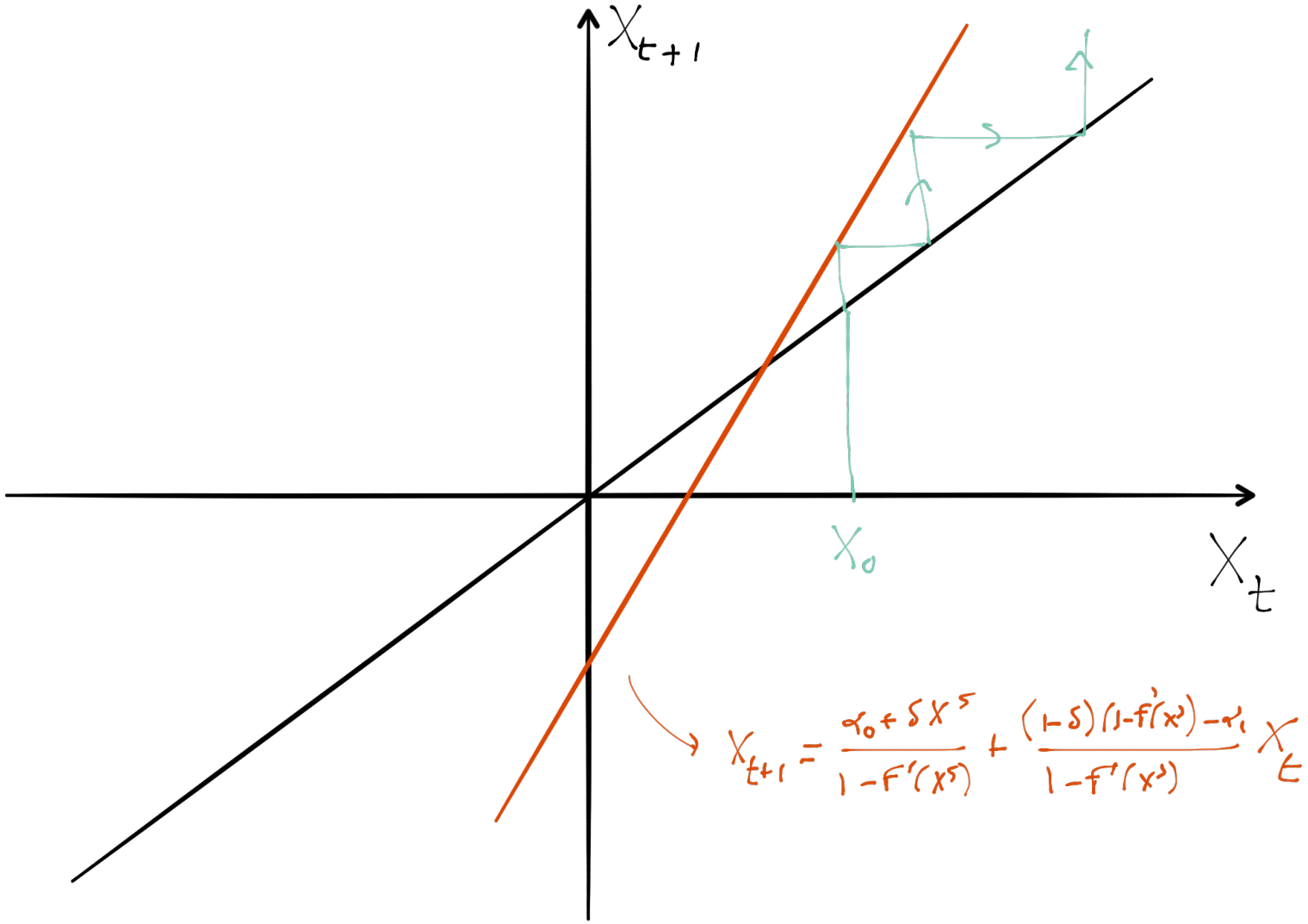
With strategic substitutability ($F'(\cdot) < 0$), the economy is always stable

Proposition 3

With strategic complementarity ($F'(\cdot) > 0$), there is always a level of complementarity smaller than 1 such that the SS is (locally) unstable.

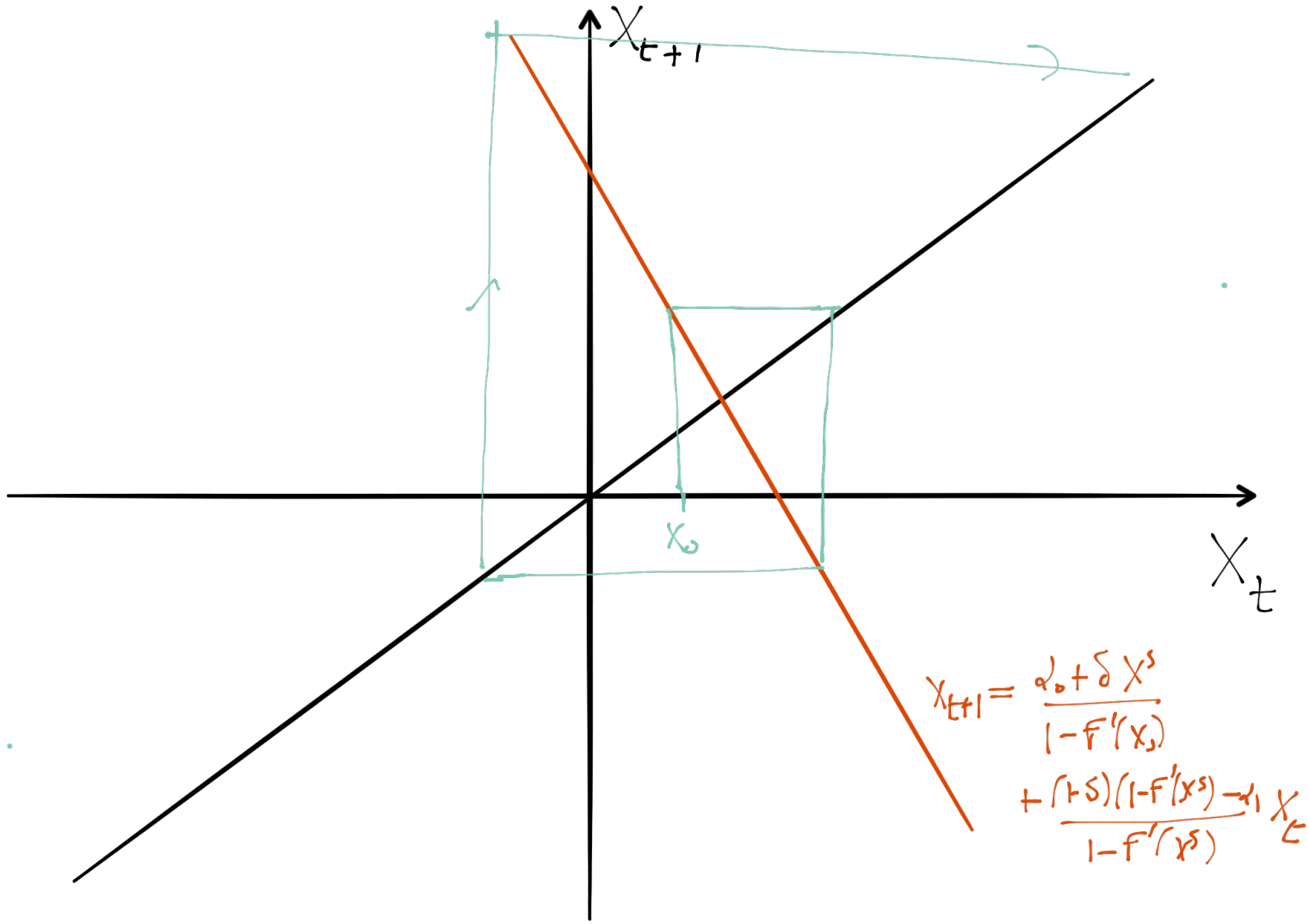
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One-Dimension Case



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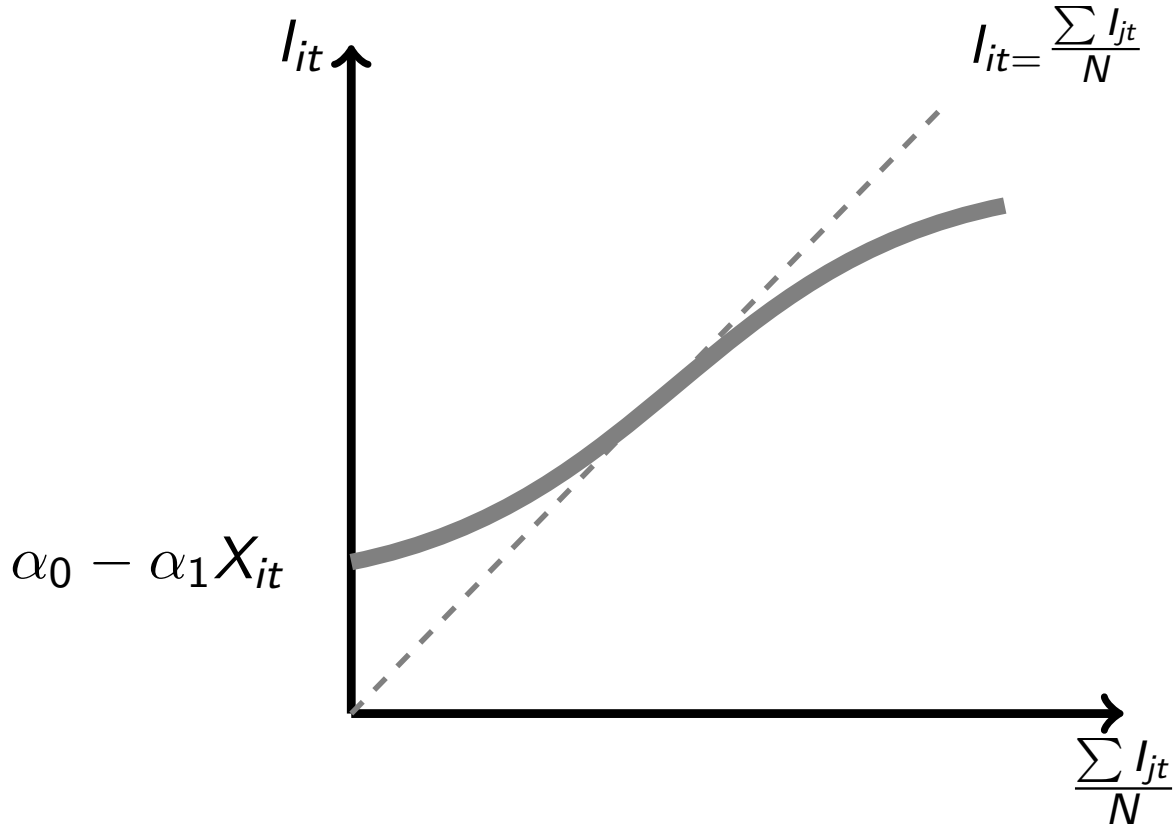
One-Dimension Case



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One-Dimension Case

- The SS X^s becomes locally unstable
- "Data" do not look explosive
- Let's assume that strategic complementarities dies out when the economy is far above of below X^s
- F is "S-shaped"



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One-Dimension Case

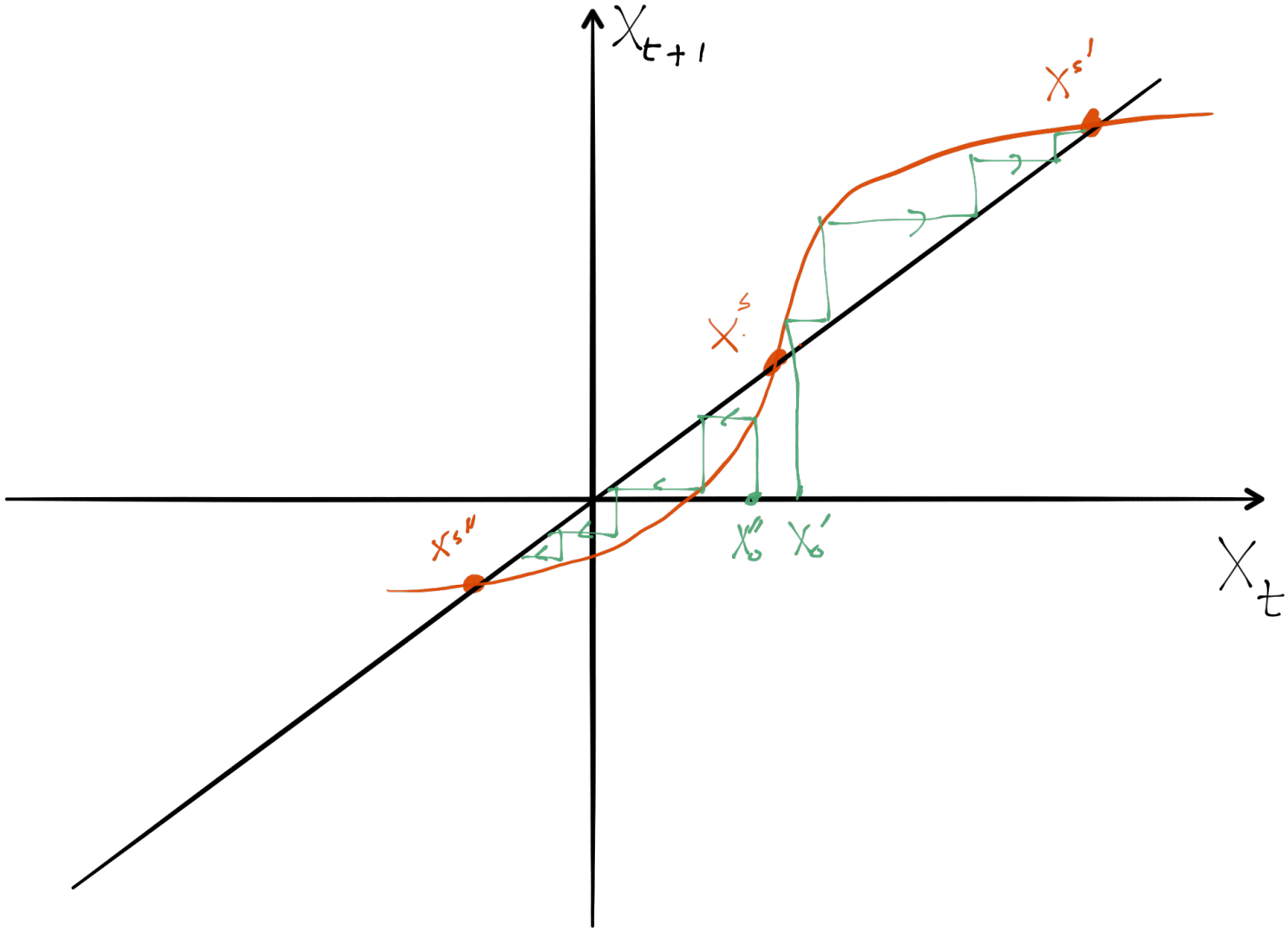
- Assume for simplicity that F is symmetric wrt X^s

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(X^s)} + \frac{((1-\delta)(1-F'(X^s)) - \alpha_1)}{1 - F'(X^s)} X_t$$

- The dynamics of the economy will very much depend on whether $\frac{((1-\delta)(1-F'(X^s))-\alpha_1)}{1-F'(X^s)}$ is positive or negative
  - × Positive : Hysteresis
  - × Negative : Limit cycle
- Need to think of environments that makes this slope is positive or negative

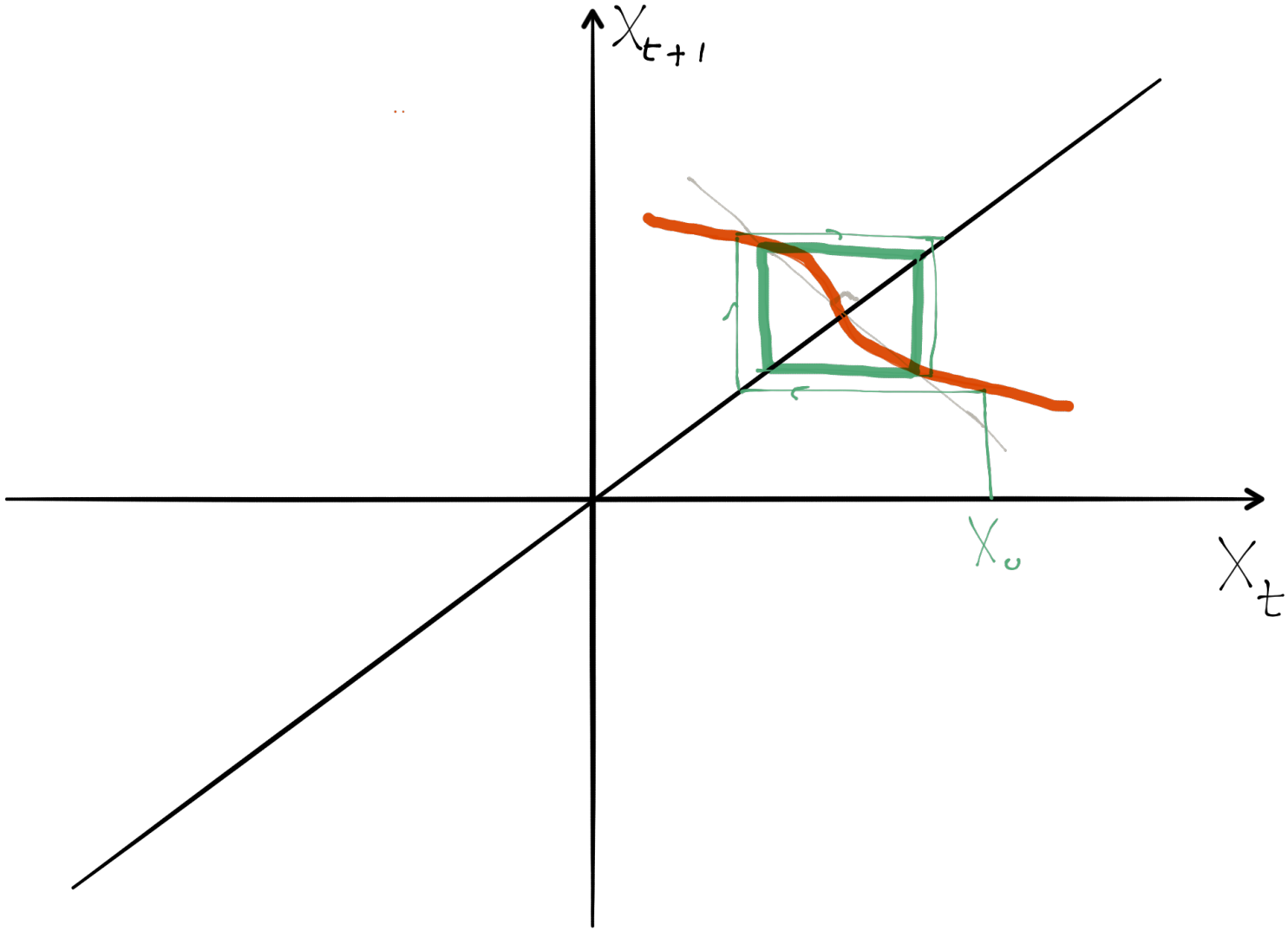
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One-Dimension Case



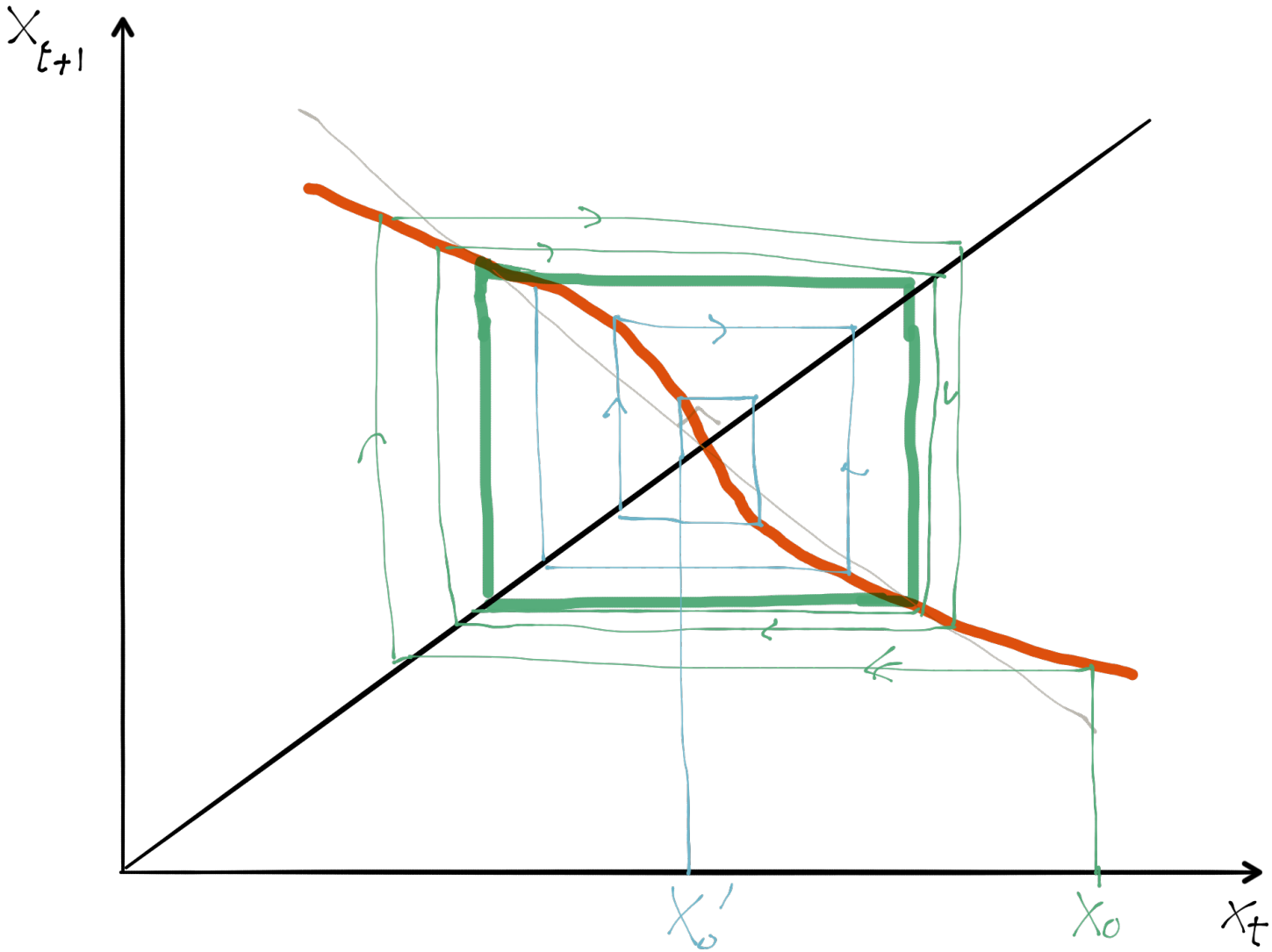
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One-Dimension Case



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One-Dimension Case



Simple Dynamics with Strategic Interactions

One-Dimension Case

- Problem with one dimension model : the dynamics does not look like a business cycle
- X_t is *not* positively correlated.
- This needs not to be true in the more general model where $\alpha_2 > 0$

Simple Dynamics with Strategic Interactions

Two-Dimension Case

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + I_{it} \\ I_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1} + F(I_t) \end{aligned}$$

- Without strategic interactions ($F(\cdot) \equiv 0$)
- When all agent behave symmetrically :

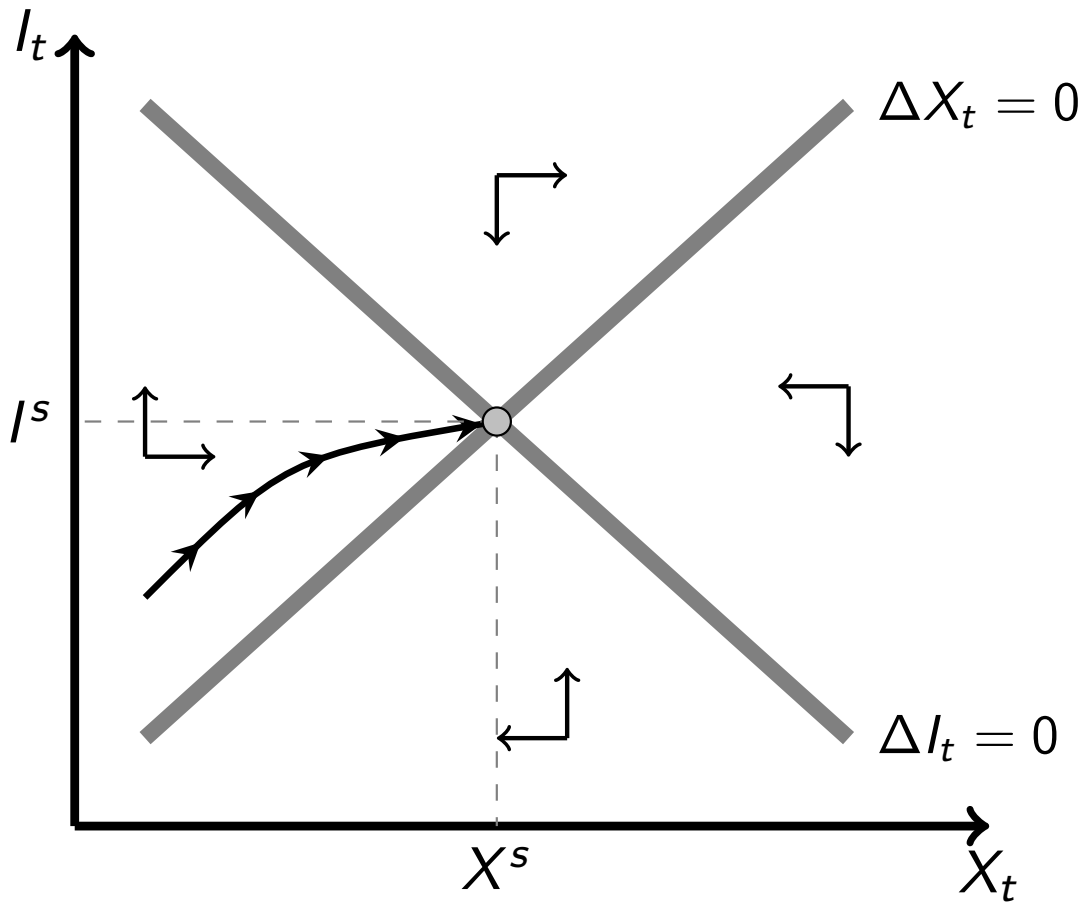
$$\begin{pmatrix} I_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_2 - \alpha_1 & -\alpha_1(1-\delta) \\ 1 & 1-\delta \end{pmatrix}}_{M_L} \begin{pmatrix} I_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}$$

Proposition 4

Both eigenvalues of the matrix M_L lie within the unit circle. Therefore, the system is stable.

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Two-Dimension Case



Here we have assumed that the two eigenvalues are real and positive.

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Two-Dimension Case

- Math
- Dynamics is

$$\begin{pmatrix} I_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_2 - \alpha_1 & -\alpha_1(1-\delta) \\ 1 & 1-\delta \end{pmatrix}}_{M_L} \begin{pmatrix} I_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_0 + F(I_t) \\ 0 \end{pmatrix}$$

- Local dynamics is

$$\begin{pmatrix} I_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\alpha_2-\alpha_1}{1-F'(I^s)} & -\frac{\alpha_1(1-\delta)}{1-F'(I^s)} \\ 1 & 1-\delta \end{pmatrix}}_{M} \begin{pmatrix} I_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \left(1 - \frac{\alpha_2-\alpha_1}{1-F'(I^s)}\right) I^s + \left(\frac{\alpha_1(1-\delta)}{1-F'(I^s)}\right) X^s \\ 0 \end{pmatrix}$$

- When $F'(I^s)$ varies from $-\infty$ to 1, eigenvalues of M vary

Proposition 5

As $F'(I^S)$ varies from 0 to $-\infty$, the eigenvalues of M always stay within the unit circle and therefore the system remains locally stable.

Proposition 6

As $F'(I^s)$ varies from 0 towards 1, the dynamic system will become locally unstable.

(bifurcation)

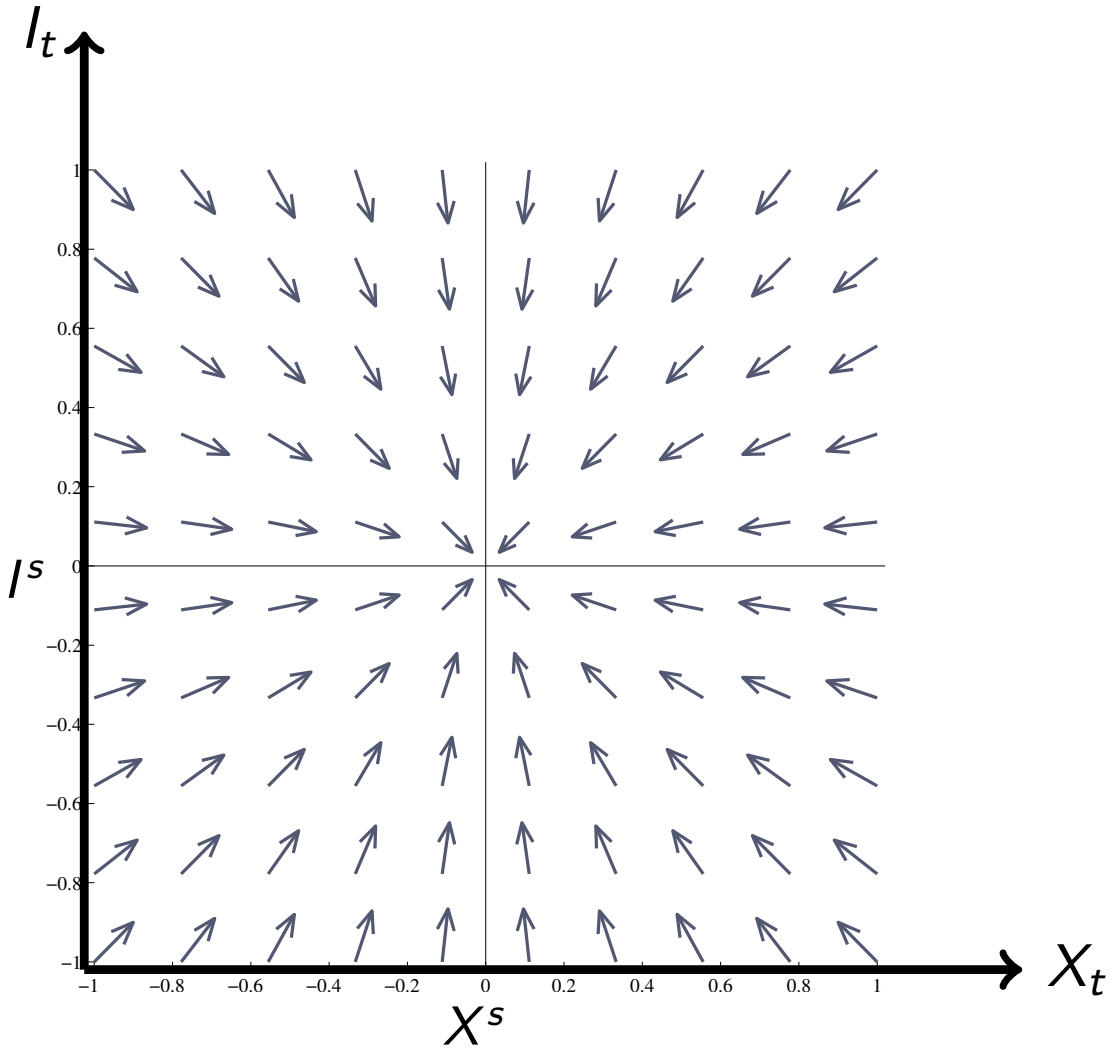
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Two-Dimension Case

- Here we can have a limit cycle with persistence
- Consider the steady state (X^s, I^s)
- Strategic complementarities : centrifugal force that pushes away from the steady state when close to.
- Accumulated variable X : centripetal force that pushes towards the steady state when away from.
- The steady state locally unstable, but forces push the economy back to the steady state when it is further from it.

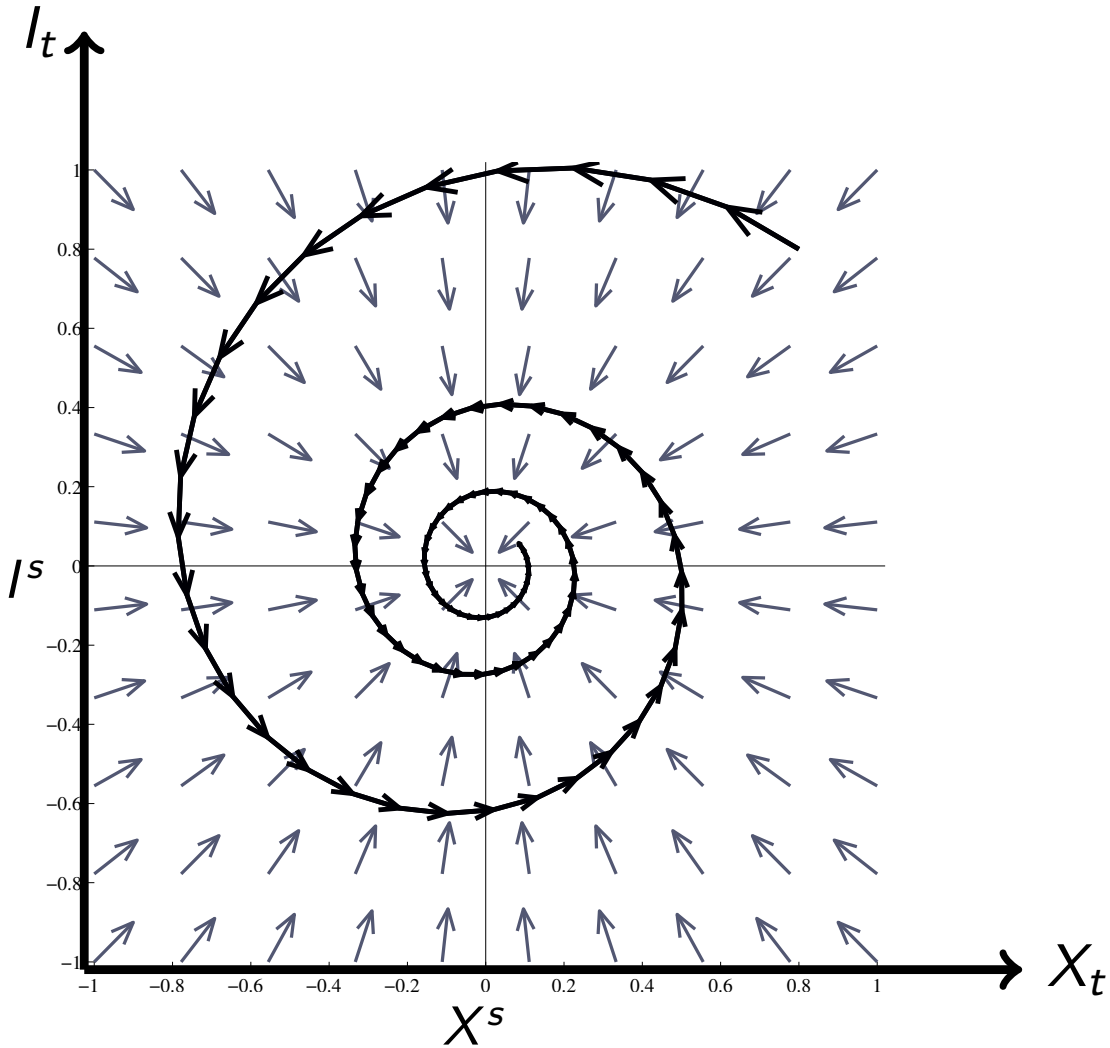
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Two-Dimension Case - Global stability



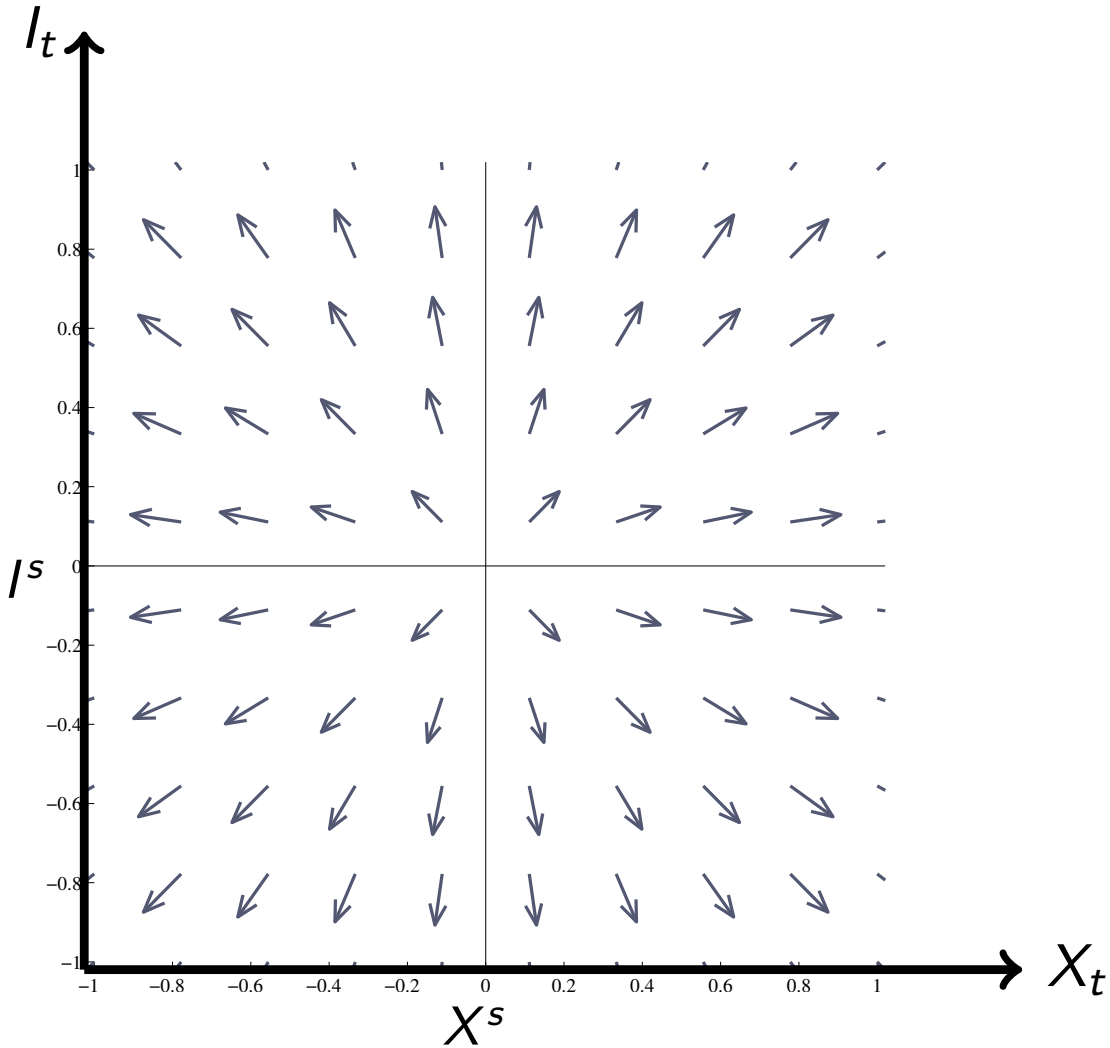
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Two-Dimension Case - Global stability



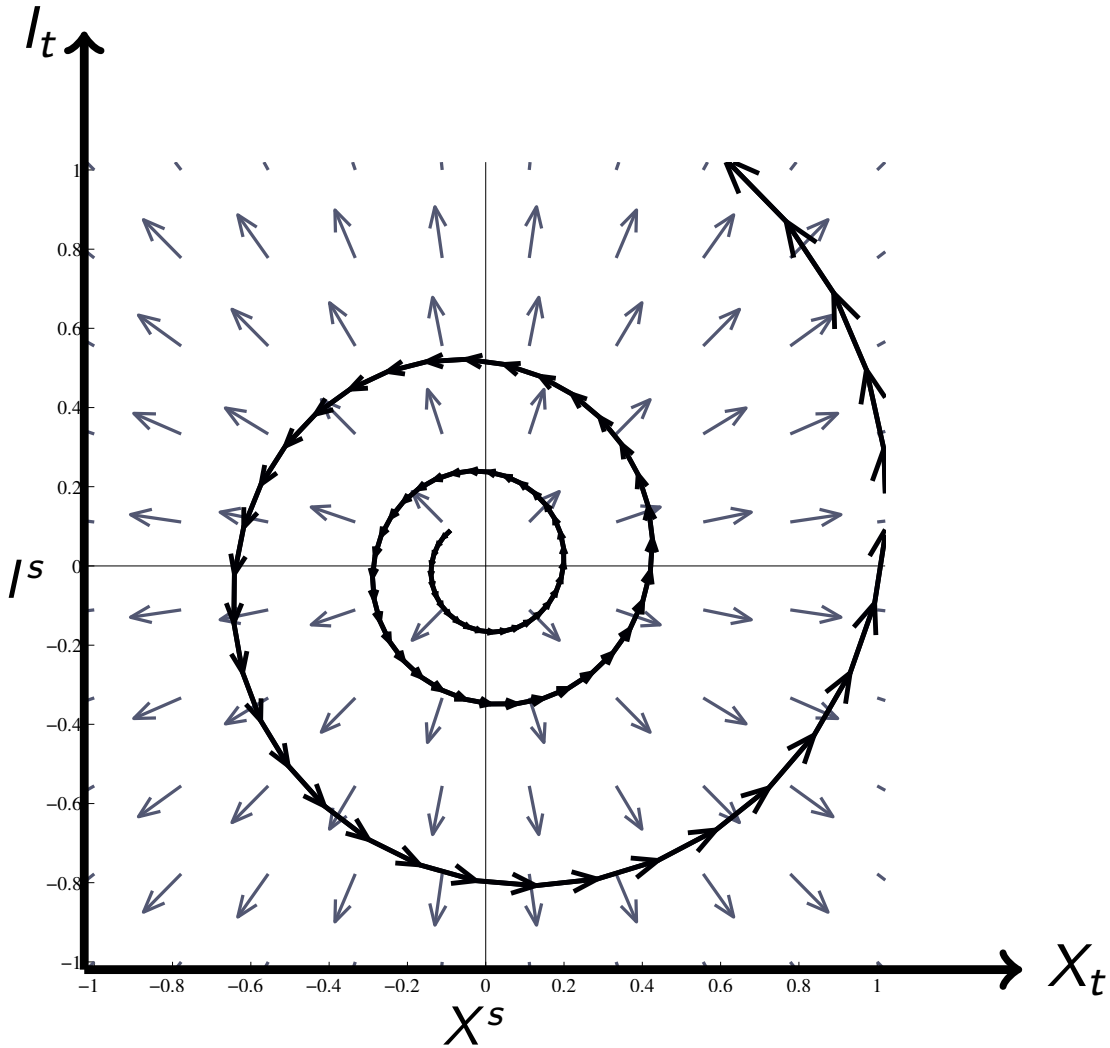
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Two-Dimension Case - Global instability



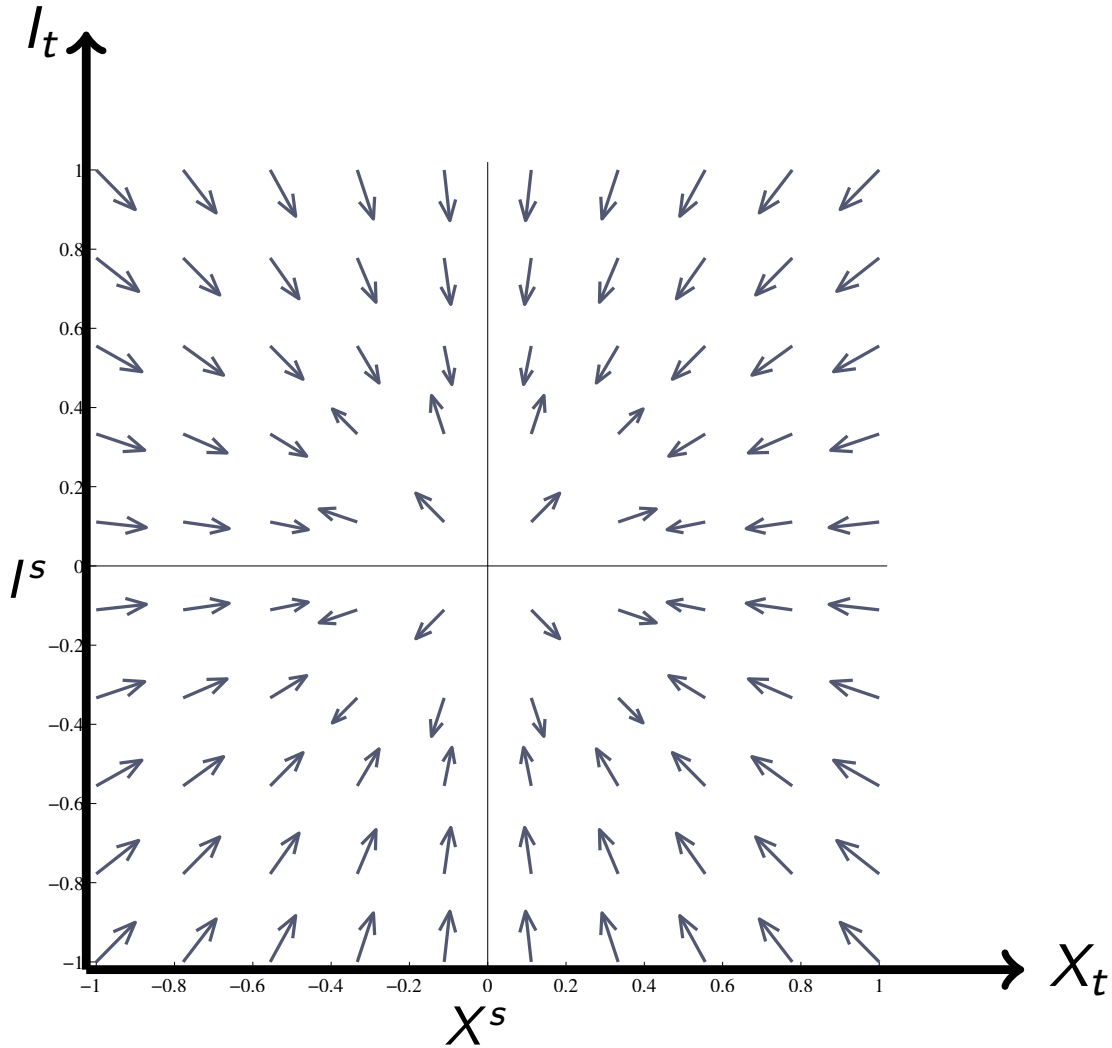
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Two-Dimension Case - Global instability



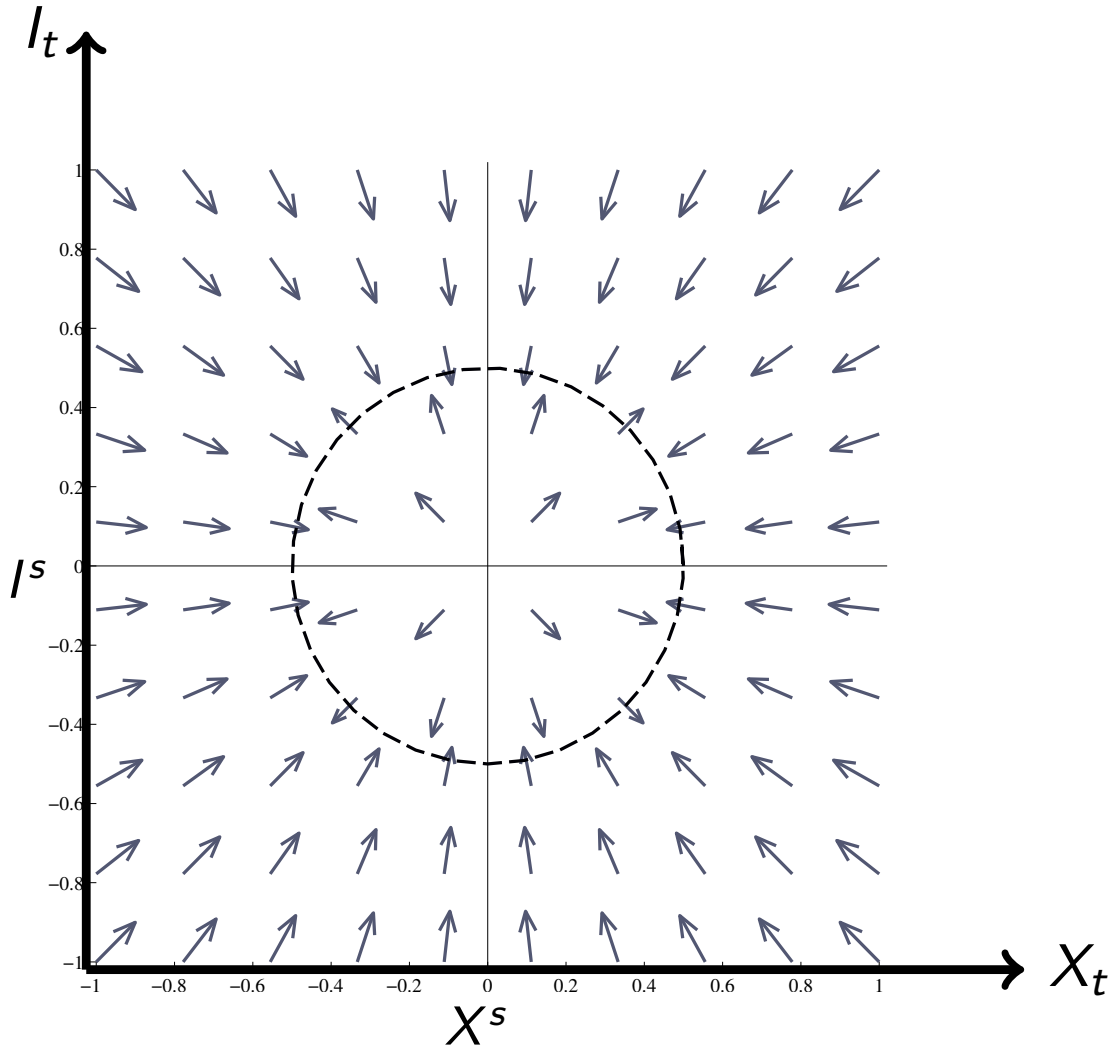
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Two-Dimension Case - Stable limit cycle



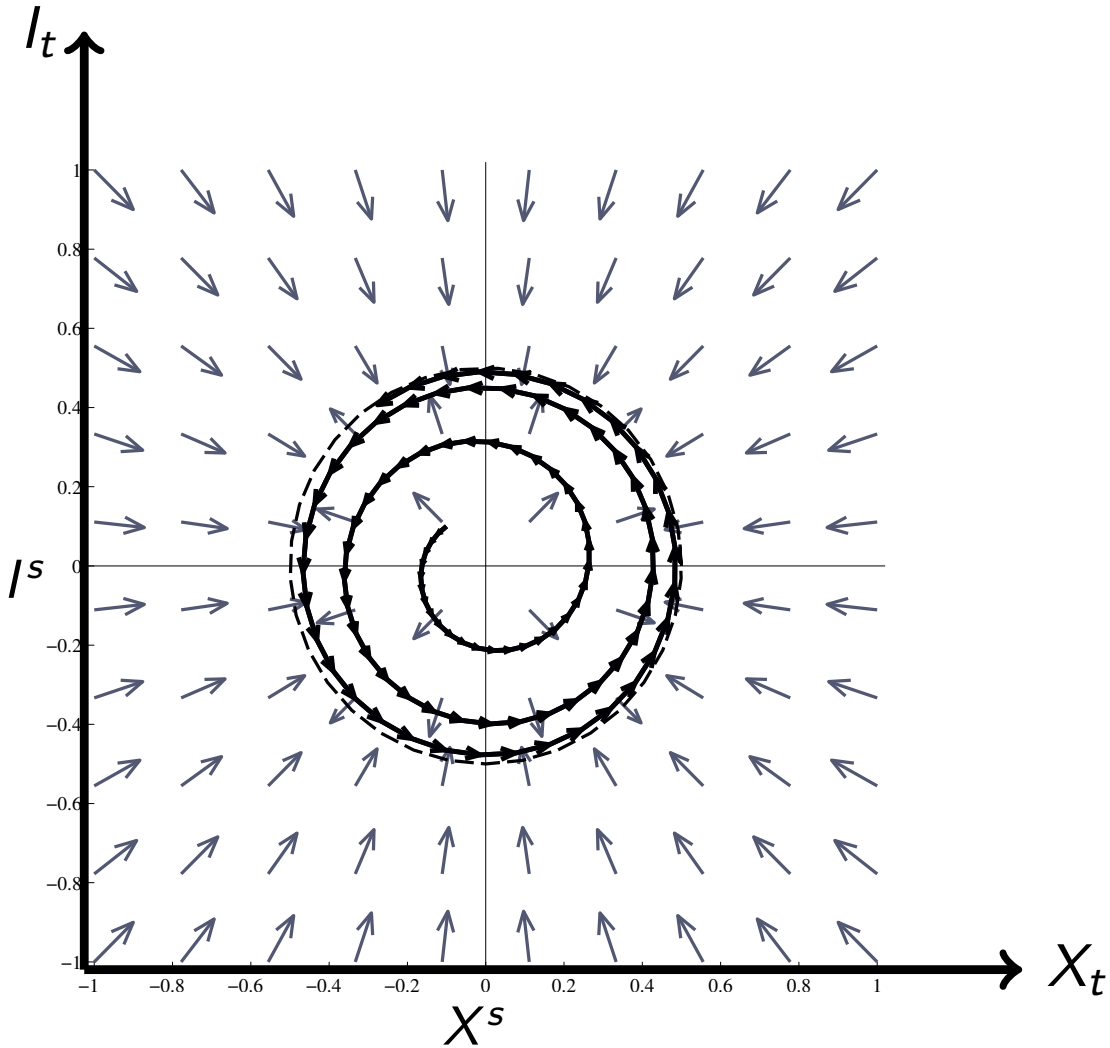
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Two-Dimension Case - Stable limit cycle



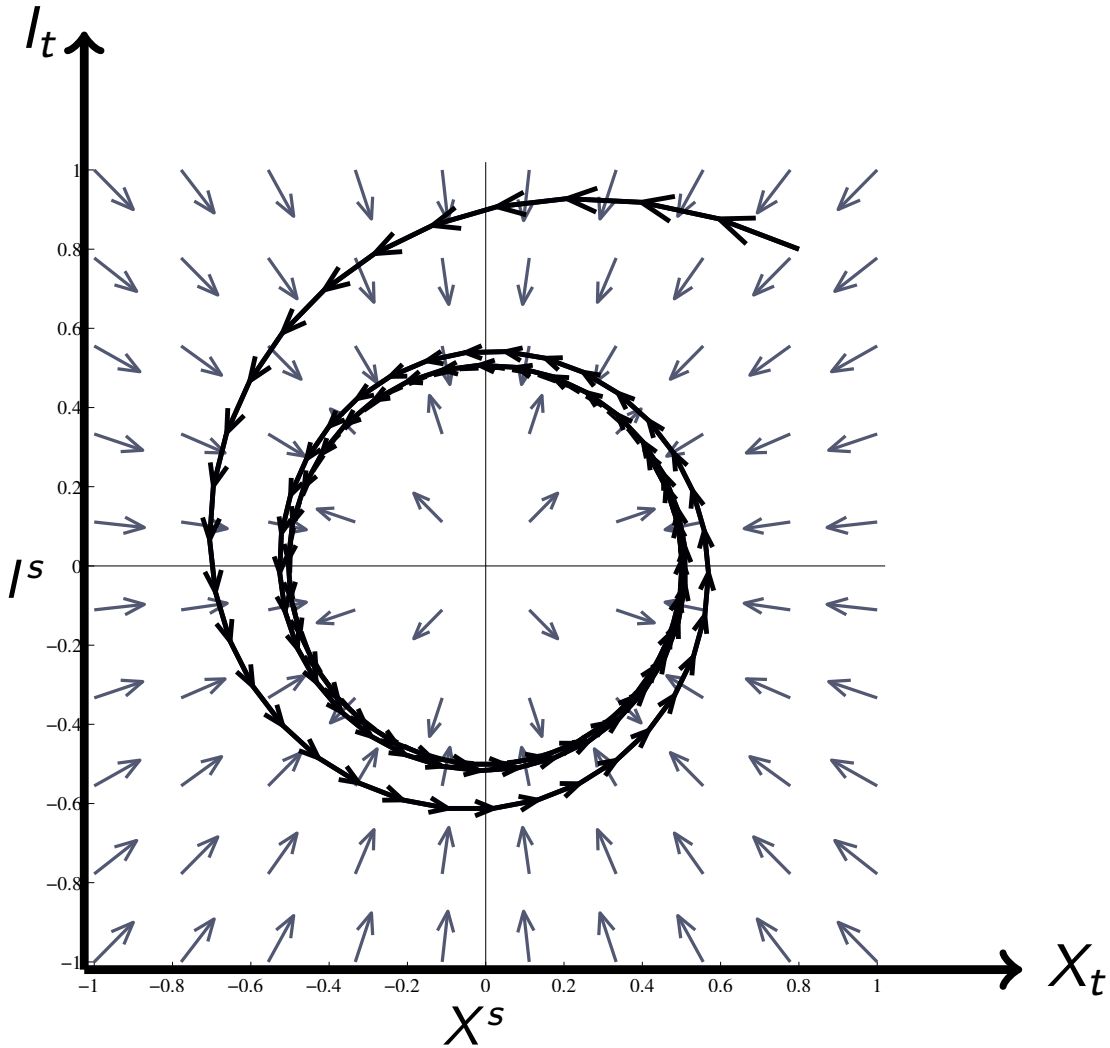
Simple Dynamics with Strategic Interactions

Two-Dimension Case - Stable limit cycle



Simple Dynamics with Strategic Interactions

Two-Dimension Case - Stable limit cycle



Simple Dynamics with Strategic Interactions

Two-Dimension Case

- Discrete time version of the HOPF theorem.
- Consider the 2-dimensional smooth system

$$X_{t+1} = F(X_t; \alpha)$$

- Assume : Steady State for $\alpha = 0$: $F(\vec{0}; 0) = 0$
- Assume : Jacobian of F at (0,0) has two eigenvalues $e^{\pm i\theta_0}$
- That system has a bifurcation at $\alpha = 0$
- *i.e.* the Steady State changes stability with α
- Then, the system $X_{t+1} = F(X_t; \alpha)$ has a limit cycle near the origin.
- Note : other possibilities is that one eigenvalue is 1 or -1.

1. Reduced form model

Looking for the limit cycle

- Nice theorem : we simply have to look at the *linearized* dynamics to prove existence of a limit cycle
- The parameter α is here the degree of strategic complementarities at the steady state $F'(I^s)$
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable

- 3 types of bifurcation
  - × Fold bifurcation : appearance of an eigenvalue equal to 1,
  - × Flip bifurcation : appearance of an eigenvalue equal to -1
  - × HOPF bifurcation : appearance of two complex conjugate eigenvalues of modulus 1.
- We are interested in HOPF bifurcation because the limit cycle will be "persistent"

Proposition 7

As $F'(I^s)$ varies from 0 towards 1, the dynamic system will become unstable and :

- *If $\alpha_2 > \alpha_1/(2-\delta)^2$, then a* Hopf *(*Neimark-Sacker*) bifurcation will occur.*
- *If $\alpha_2 < \alpha_1/(2-\delta)^2$, then a flip bifurcation will occur.*

- In the case of flip and HOPF bifurcation, a limit cycle appears
- In case of HOPF, the cycle is "smooth"
- Condition for an HOPF bifurcation :

$$I_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1} + F\left(\frac{\sum I_{jt}}{N}\right)$$

  - × as δ approaches 0, $\alpha_2 > 1/4$
  - × in general, what is needed is δ not too large and α_2 large enough

- In the case of the HOPF bifurcation, the limit cycle can be attractive (*the bifurcation is supercritical*) or repulsive (*the bifurcation is subcritical*)

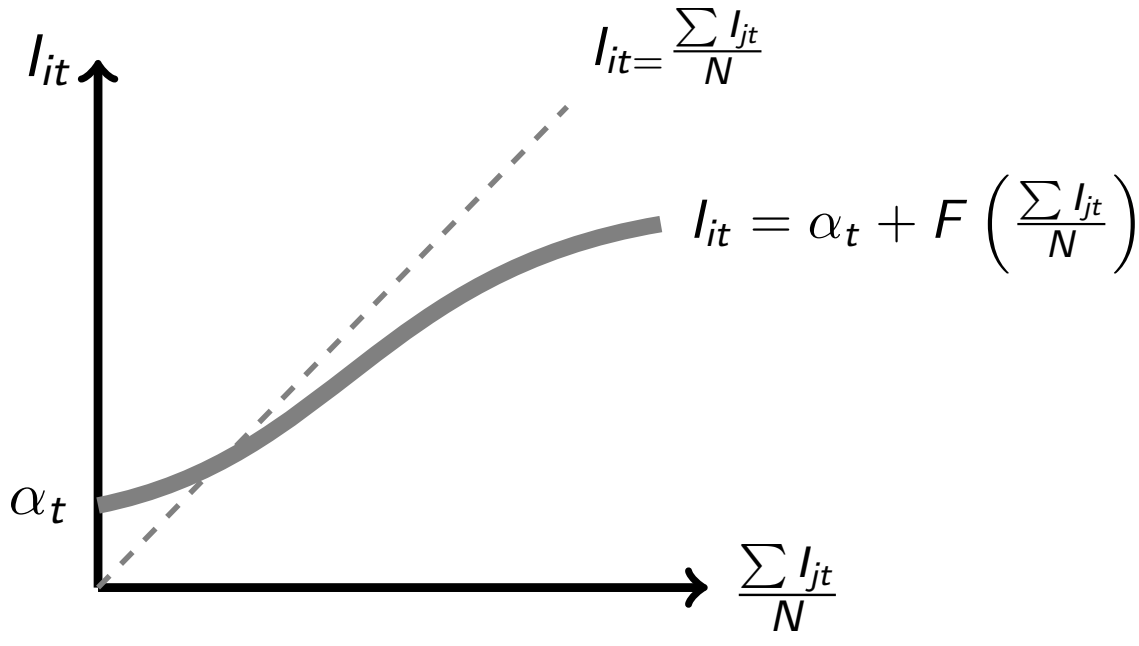
1. Reduced form model

Stability of the limit cycle

- Proposition 8

  If $F'''(I^s)$ is sufficiency negative, then the HOPF *bifurcation will be supercritical. Therefore, the limit cycle is attractive.*

  - $F''' < 0$ corresponds to an $S-$*shaped* reaction function



  -

$$\alpha_t = \alpha_0 - \alpha_1 X_{it} + \alpha_2 I_{it-1}$$