FRANCK PORTIER
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M1 MACRO

GROUP A

Session 13

These notes are written during class. They are posted "for the record" and may contain mistakes; they are not intented to be read by anyone that did not attend the class.

Dynamic Labor Demand model

- $Y = F(L) = \alpha_0 L_t - \frac{\alpha_1}{2} L_t^2$

  $F'(L) = \alpha_0 - \alpha_1 L_t > 0$ because α_0 is "large enough"

  $F'' = -\alpha_1 < 0$

Adj.

- Cost func: $C(\Delta L_t) = \frac{b}{2}(L_t - L_{t-1})^2$

  $\Delta L_t = L_t - L_{t-1}$

- W_t = wage

Firm problem - Price of production = 1

$$\underset{\{L_{t+i}\}}{\text{Max}} \; V_t = \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} \Pi_{t+i}$$

$$\Pi_{t+i} = \text{profit} = F(L_{t+i}) - W_{t+i} L_{t+i} - \varepsilon\left(L_{t+i} - L_{t+i-1}\right)$$

$$V_t = \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i \times \left[\alpha_0 L_{t+i} - \frac{\alpha_1}{2} L_{t+i}^2 - W_{t+i} L_{t+i} - \frac{b}{2}\left(L_{t+i} - L_{t+i-1}\right)^2 \right]$$

$$\max \; U_t = \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i \left(\alpha_0 L_{t+i} - \frac{\alpha_1}{2} L_{t+i}^2 - W_{t+i} L_{t+i} - \frac{b}{2}\left(L_{t+i} - L_{t+i-1}\right)^2 \right)$$

$$\text{w.r.t } \{L_{t+i}\}_{i=0}^{\infty}$$

$$\text{foc } L_{t+i} : \left(\frac{1}{1+r}\right)^i \left(\alpha_0 - \alpha_1 L_{t+i} - W_{t+i} - b\left(L_{t+i} - L_{t+i-1}\right)\right)$$

$$+ \left(\frac{1}{1+r}\right)^{i+1} b\left(L_{t+i+1} - L_{t+i}\right) = 0$$

$\rightarrow$ I look at $i=0$ (for convenience) $\forall i \geq 0$

FOC

$$\alpha_0 - \alpha_1 L_t - W_t - b(L_t - L_{t-1}) + \frac{1}{1+r} b(L_{t+1} - L_t) = 0$$

$$= \beta$$

- If $b = 0$: No adjustment cost

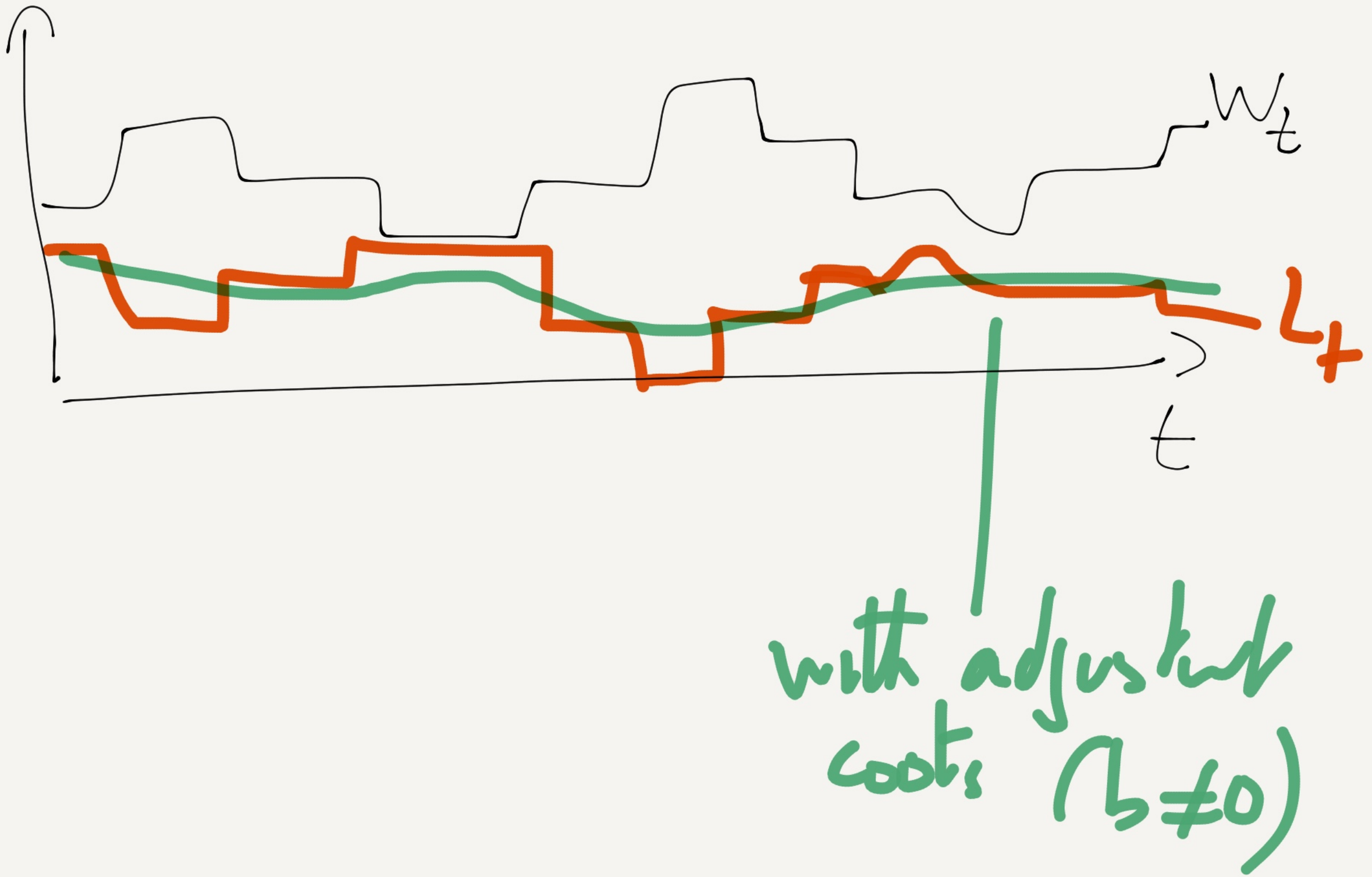
$$\alpha_0 - \alpha_1 L_T - W_t = 0$$

optimal choice → optimal size

$$L_t = \frac{\alpha_0 - W_t}{\alpha_1}$$

(assuming $\alpha_0 - W_t > 0$)

STATIC CHOICE


W_t
L_t
t
with adjustment
costs ($b \neq 0$)

. When $b \neq 0$

(1) $a_0 - a_1 L_t - W_t - b(L_t - L_{t-1}) + \beta b(L_{t+1} - L_t) = 0$

So long the model = finding L_t at every $t \geq 0$
. given L_{-1}
such that (1) is satisfied.

Math Property

- $aX_{t+1} + bX_t + cX_{t-1} = 0 \quad (*)$

characteristic polynomial of this equation.

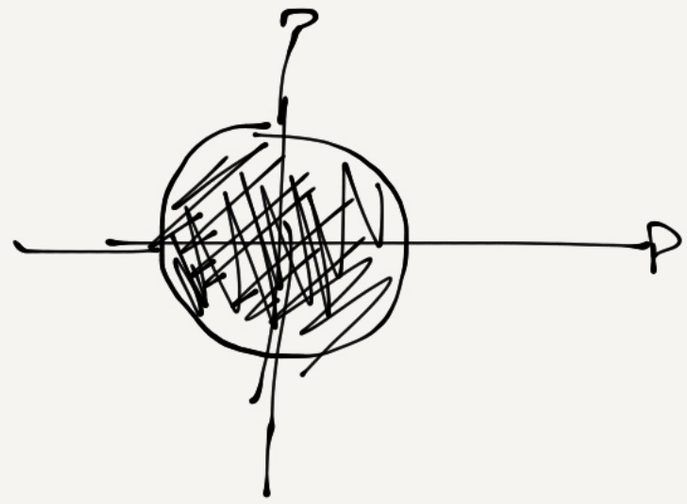
$$P(\lambda) = a\lambda^2 + b\lambda + c = 0$$

let λ_1, λ_2 be the two roots of that polynomial
(real or complex)

↳ meaning: $P(\lambda_1) = 0$
$P(\lambda_2) = 0$

the solution of (*) writes

$$X_t = c_1 \lambda_1^t + c_2 \lambda_2^t$$



where c_1 and c_2 are determined by 2 boundary conditions.

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for ex: if X_0, X_1 are given, then

2eq in (c_1, c_2) $\leftarrow$

$$\begin{cases} X_0 = c_1 \lambda_1^0 + c_2 \lambda_2^0 = c_1 + c_2 \\ X_1 = c_1 \lambda_1 + c_2 \lambda_2 \end{cases}$$

(1) $\alpha_0 - \alpha_1 L_t - w_t - b(L_t - L_{t-1}) + \beta b(L_{t+1} - L_t) = 0$

- **Steady state**

  - Assume $w_t = \bar{w}$ $\forall t$

  - $\bar{L}$ is the solution of

$$\alpha_0 - \alpha_1 \bar{L} - \bar{w} - b(\bar{L} - \bar{L}) + \beta b(\bar{L} - \bar{L}) = 0$$

↳ (1) when $L_t = \bar{L}$ $\forall t$

$$\boxed{\bar{L} = \frac{\alpha_0 - \bar{w}}{\alpha_1}}$$

. Define $\hat{L}_t = L_t - \bar{L}$

(1) $\alpha_0 - \alpha_1\left(\overbrace{L_t - \bar{L} + \bar{L}}^{\hat{L}_t}\right) - \left(W_t - \overline{W}\right) - b\left(\overbrace{L_t - \bar{L} + \bar{L}}^{\hat{L}_t} - \left(\overbrace{L_{t-1} - \bar{L} + \bar{L}}^{\hat{L}_{t-1}}\right)\right)$

$+\beta b\left(\underbrace{L_{t+1} - \bar{L}}_{\hat{L}_{t+1}} + \bar{L} - \left(\underbrace{L_t - \bar{L}}_{\hat{L}_t} + \bar{L}\right)\right) = 0$

$\Leftrightarrow$

$\alpha_0 - \alpha_1 \hat{L}_t - \alpha_1 \bar{L} - \hat{W}_t - \overline{W} - b\left(\hat{L}_t - \hat{L}_{t-1}\right) - b\left(\underbrace{\bar{L} - \bar{L}}_{0}\right)$

$+\beta b\left(\hat{L}_{t+1} - \hat{L}_t\right) + \beta b\left(\underbrace{\bar{L} - \bar{L}}_{0}\right) = 0$

$\alpha_0 - \alpha_1 \bar{L} - \overline{W} = \alpha_0 - \alpha_1\left(\frac{\alpha_0 - \overline{W}}{\alpha_1}\right) - \overline{W} = 0$

Therefore

$$(1) \Leftrightarrow -\alpha_1 \widehat{\Gamma}_t - \widehat{w}_t - b\left(\widehat{\Gamma}_t - \widehat{\Gamma}_{t-1}\right) + \beta b\left(\widehat{\Gamma}_{t+1} - \widehat{\Gamma}_t\right) = 0$$

$$\Leftrightarrow \boxed{\beta b\, \widehat{\Gamma}_{t+1} - (\alpha_1 + b + \beta b)\, \widehat{\Gamma}_t + b\, \widehat{\Gamma}_{t-1} = \widehat{w}_t}$$

Assume $\widehat{w}_t = 0 \ \forall t \Rightarrow w_t = \bar{w}$

$$\beta b\, \widehat{\Gamma}_{t+1} - (\alpha_1 + b + \beta b)\, \widehat{\Gamma}_t + b\, \widehat{\Gamma}_{t-1} = 0$$

Solve $\beta b \bar{L}_{t+1} - (\alpha_1 + b + \beta b)\bar{L}_t + b\bar{L}_{t-1} = 0$

Characteristic polynomial:

$$P(\lambda) = \beta b \lambda^2 - (\alpha_1 + b + \beta b)\lambda + b = 0$$

λ_1, λ_2 are the two roots

Properties of polynomials if $P(\lambda) = a\lambda^2 + b\lambda + c$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_1 + \lambda_2 = -\frac{b}{a} \quad , \quad \lambda_1 \times \lambda_2 = \frac{c}{a}$$

Here $\lambda_1 \times \lambda_2 = \frac{1}{\beta} > 1$ (because $\beta = \frac{1}{1+\alpha} < 1$)

$$\lambda_1 + \lambda_2 = \frac{\alpha_1 + b + \beta b}{\beta b} > 0$$

$\lambda_1 \lambda_2 > 0 \Rightarrow \lambda_1$ and λ_2 have the same sign

$\lambda_1 + \lambda_2 > 0 \Rightarrow \lambda_1$ and λ_2 are both positive

$\lambda_1 \lambda_2 > 1 \Rightarrow \lambda_1$ and/or λ_2 is > 1

$\rightarrow$ say it is λ_2

$P(1) = -\alpha_1 < 0$

$P(\lambda)$? $\cup$ → because $\beta b > 0$



$\Rightarrow$ proves that $\lambda_1 \in [0, 1[$

Therefore, the solution is

$$\Sigma_t = c_1 \lambda_1^t + c_2 \lambda_2^t$$

for some c_1, c_2 to be found.