2017-2018 – DEEQA – TSE – Advanced Macroeconomics

Lecture 3 : Business Cycle Models : The Current Consensus

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Disclaimer

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1.Overview of Consensus

- The object of interest to macroeconomic researchers is to provide an explanation to business cycle fluctuations and to use the explanation to help guide policy.
- An explanation(theory) is composed of two elements : a list of the exogenous factors that perturb the system, and a description of the mechanism by which external factors influence internal (endogenous) factors. (driving forces may be observable or unobservable)
- The "main" tradition in economics is to search for an explanation to a set of observations as resulting from the interaction of agents trying to solve an allocation problem subject to a set of constraints (physical, institutional, informational). The interaction (or equilibrium) is generally modeled as a fixed point.

- For us, this implies trying to explain recurrent business cycle phenomena (positive co-movements between employment, output, consumption, investment) as resulting form some type of time varying allocation problem. But what is this allocation problem ? ?
- Note : This is not the only way of approaching the problem (ex : "Agent-Based Models" with mechanistic interaction of set behaviors).

- Two possibilities : either the gains from trade vary over time, or our capacity to exploit these gains from trade vary (Or both).
- Real business cycle theory suggest the first may offer an answer, monetarism basically suggest the second, and the new Keynesien model is a particular combination of the two.
- Many "Keynesiens" like instead to state the issue as one of variation in aggregate demand (but what does this mean ? ?). Simple example of equilibrium thinking in the current slow expansion : does supply matter. Look at correlation between employment growth and growth of labour age population....

2. The Standard Real Business Cycle (RBC) Model

In A Nutshell

- Kydland and Prescott suggested that business cycles may reflect the fact that gains from trade do not grow smoothly over time, but instead may be stochastic. More to the point, they showed that the optimal adjustment to stochastic technology growth would have many of the features of business cycles.
- As this was very controversial, they presented a very simple calibration exercise (tying their hands) to illustrate the strength of the idea.
- Took as exogenous process a measure of Solow residual
- $\Delta A_t = \Delta \frac{Y_t}{L_t} - \alpha \Delta \frac{K_t}{L_t}$
- Feed it into a simple representative agent model $(U(C, l) = \ln C_t + v(1 - l_t)^{1-\gamma}, \quad Y_t = A_t K^{\alpha} L_t^{1-\alpha})$.
- If the γ sufficiently small, it reproduces business cycle statistics (Co-movements and volatility)

- Perfectly competitive economy
- Optimal growth model + Labor decisions
- 2 types of agents : Households and Firms
- Shocks to productivity
- Pareto optimal economy
- We will solve for the competitive equilibrium

- Mass of agents $= 1$ (no population growth)
- Identical agents $+$ All face the same aggregate shocks (no idiosyncratic uncertainty)
- $\rightsquigarrow$ One representative household

- Infinitely lived rational agent with intertemporal utility

$$E_t \sum_{s=0}^{\infty} \beta^s U_{t+s}$$

  $\beta \in (0,1)$: discount factor,
- Preferences over
  - × a consumption bundle
  - × leisure

- $U_t = U(C_t, \ell_t)$ with $U(\cdot, \cdot)$
  - × class $\mathcal{C}^2$, strictly increasing, concave and satisfy Inada conditions
  - × compatible with balanced growth :

$$U(C_t, \ell_t) = \begin{cases} \frac{C_t^{1-\sigma}}{1-\sigma} v(\ell_t) & \text{for } \sigma \in \mathbb{R}^+ \backslash \{1\} \\ \text{or} & \\ \log(C_t) + v(\ell_t) & \end{cases}$$

  - × Preferences are therefore given by

$$E_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}, \ell_{t+s}) \right]$$

- Household faces three constraints
- Time constraint

$$h_{t+s} + \ell_{t+s} \leqslant T = 1$$

(for convenience T=1)

- Budget constraint

$$\underbrace{B_{t+s}}_{\text{Purchases of bonds}} + \underbrace{C_{t+s} + I_{t+s}}_{\text{Expenditures on goods}} \leqslant \underbrace{(1 + r_{t+s-1})B_{t+s-1}}_{\text{Financial income from bonds}} + \underbrace{W_{t+s}h_{t+s}}_{\text{Labor income}} + \underbrace{z_{t+s}K_{t+s}}_{\text{Capital income}}$$

- Capital Accumulation

$$K_{t+s+1} = I_{t+s} + (1 - \delta)K_{t+s}$$

$\delta \in (0, 1)$: depreciation rate

- The household decides on consumption, labor, leisure, investment, bond holdings and capital formation maximizing utility constraint, taking the constraints into account

$$\max_{\{C_{t+s},h_{t+s},\ell_{t+s},I_{t+s},K_{t+s+1},B_{t+s}\}_{s=0}^{\infty}} E_t\left[\sum_{s=0}^{\infty}\beta^s U(C_{t+s},\ell_{t+s})\right]$$

subject to the sequence of constraints

$$\begin{cases} h_{t+s}+\ell_{t+s}\leqslant 1 \\ B_{t+s}+C_{t+s}+I_{t+s}\leqslant (1+r_{t+s-1})B_{t+s-1}+W_{t+s}h_{t+s}+z_{t+s}K_{t+s} \\ K_{t+s+1}=I_{t+s}+(1-\delta)K_{t+s} \\ K_t,\ B_{t-1} \text{ given} \end{cases}$$

$$\max_{\{C_{t+s},h_{t+s},K_{t+s+1},B_{t+s}\}_{s=0}^{\infty}} E_t\left[\sum_{s=0}^{\infty}\beta^s U(C_{t+s},1-h_{t+s})\right]$$

subject to

$$B_{t+s}+C_{t+s}+K_{t+s+1}\leqslant(1+r_{t+s-1})B_{t+s-1}+W_{t+s}h_{t+s}+(z_{t+s}+1-\delta)K_{t+s}$$

- Write the Lagrangian

$$\mathcal{L}_t = E_t\sum_{s=0}^{\infty}\beta^s\Bigg[U(C_{t+s},1-h_{t+s})+\Lambda_{t+s}\Big((1+r_{t+s-1})B_{t+s-1}+ \\ +W_{t+s}h_{t+s}+(z_{t+s}+1-\delta)K_{t+s}-C_{t+s}-B_{t+s}-K_{t+s+1}\Big)\Bigg]$$

- First order conditions ($\forall\ s \geq 0$)

$$\begin{array}{lcl} C_{t+s} & : & E_t U_c(C_{t+s}, 1-h_{t+s}) = E_t \Lambda_{t+s} \\ h_{t+s} & : & E_t U_\ell(C_{t+s}, 1-h_{t+s}) = E_t(\Lambda_{t+s} W_{t+s}) \\ B_{t+s} & : & E_t \Lambda_{t+s} = \beta E_t((1+r_{t+s})\Lambda_{t+s+1}) \\ K_{t+s+1} & : & E_t \Lambda_{t+s} = \beta E_t(\Lambda_{t+s+1}(z_{t+s+1}+1-\delta)) \end{array}$$

and the transversality condition

$$\lim_{s \longrightarrow +\infty} \beta^s E_t \Lambda_{t+s}(B_{t+s} + K_{t+s+1}) = 0$$

- **Remark :**

  - It is convenient to write and interpret FOC for $s = 0$:

$$\begin{array}{lcl} h_t & : & U_\ell(C_t, 1-h_t) = U_c(C_t, 1-h_t)W_t \\ B_t & : & U_c(C_t, 1-h_t) = \beta((1+r_t)E_t U_c(C_{t+1}, 1-h_{t+1})) \\ K_{t+1} & : & U_c(C_t, 1-h_t) = \beta E_t(U_c(C_{t+1}, 1-h_{t+1})(z_{t+1}+1-\delta)) \end{array}$$

  and the transversality condition

$$\lim_{s \longrightarrow +\infty} E_t \beta^s U_c(C_{t+s}, 1-h_{t+s})\left(K_{t+s+1} + B_{t+s}\right) = 0$$

- Assume $U(C_t, \ell_t) = \log(C_t) + \theta \log(1 - h_t)$

$$\begin{array}{lcl} h_{t+s} & : & E_t \frac{\theta}{1-h_{t+s}} = E_t \frac{W_{t+s}}{C_{t+s}} \\ B_{t+s} & : & E_t \frac{1}{C_{t+s}} = \beta E_t (1 + r_{t+s}) \frac{1}{C_{t+s+1}} \\ K_{t+s+1} & : & E_t \frac{1}{C_{t+s}} = \beta E_t \frac{1}{C_{t+s+1}} (z_{t+s+1} + 1 - \delta) \end{array}$$

and the transversality condition

$$\lim_{s \longrightarrow +\infty} E_t \beta^s \frac{K_{t+s+1} + B_{t+s}}{C_{t+s}} = 0$$

- Mass of firms $= 1$
- Identical firms $+$ All face the same aggregate shocks (no idiosyncratic uncertainty)

$\rightsquigarrow$ Representative firm

- Produce an homogenous good that is consumed or invested
- by means of capital and labor
- Constant returns to scale technology

$$Y_t = A_t F(K_t, \Gamma_t h_t)$$

- $\Gamma_t = \gamma \Gamma_{t-1}$ Harrod neutral technological progress ($\gamma \geqslant 1$), A_t stationary (does not explain growth)
- A_t are shocks to technology. AR(1) exogenous process

$$\log(A_t) = \rho \log(A_{t-1}) + (1 - \rho) \log(\overline{A}) + \varepsilon_t$$

with $\varepsilon_t \rightsquigarrow N(0, \sigma^2)$.

- The firm decides on production plan maximizing profits

$$\max_{\{K_t, h_t\}} A_t F(K_t, \Gamma_t h_t) - W_t h_t - z_t K_t$$

- First order conditions :

$$\begin{aligned} K_t &: A_t F_K(K_t, \Gamma_t h_t) = z_t \\ h_t &: A_t F_h(K_t, \Gamma_t h_t) = W_t \end{aligned}$$

- Important : zero profits

- Cobb–Douglas production function

$$Y_t = A_t K_t^{\alpha} (\Gamma_t h_t)^{1-\alpha}$$

- First order conditions

$$\begin{aligned} K_t &: \alpha Y_t / K_t = z_t \\ h_t &: (1-\alpha) Y_t / h_t = W_t \end{aligned}$$

- The (RBC) Model Equilibrium is given by the following equations ($\forall\ t \geq 0$) :
  1. Exogenous Processes : $\log(A_t) = \rho \log(A_{t-1}) + (1-\rho)\log(\overline{A}) + \varepsilon_t$ and $\Gamma_t = \gamma \Gamma_{t-1}$
  2. Law of motion of Capital : $K_{t+1} = I_t + (1-\delta)K_t$
  3. Bond market equilibrium : $B_t = 0$
  4. Good Markets equilibrium : $Y_t = C_t + I_t$
  5. Labor market equilibrium : $\frac{U_\ell(C_t, 1-h_t)}{U_c(C_t, \ell_t)} = A_t F_h(K_t, \Gamma_t h_t)$
  6. Consumption/saving decision + Capital market equilibrium :

$$U_c(C_t, 1-h_t) = \beta E_t \left[U_c(C_{t+1}, 1-h_{t+1})(A_{t+1} F_K(K_{t+1}, \Gamma_{t+1} h_{t+1}) + 1 - \delta) \right]$$

  7. Financial markets :

$$(1 + r_t) = \frac{U_c(C_t, 1-h_t)}{\beta E_t U_c(C_{t+1}, 1-h_{t+1})}$$

- $U(C_t, \ell_t) = \log(C_t) + \theta \log(\ell_t), \qquad Y_t = A_t K_t^{\alpha} (\Gamma_t h_t)^{1-\alpha}$
- Equilibrium

$$\frac{\theta C_t}{1 - h_t} = (1 - \alpha)\frac{Y_t}{h_t}$$
$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \right) \right]$$
$$K_{t+1} = Y_t - C_t + (1 - \delta) K_t$$
$$Y_t = A_t K_t^{\alpha} (\Gamma_t h_t)^{1-\alpha}$$
$$\lim_{s \longrightarrow \infty} \beta^s E_t \left[\frac{K_{t+1+s}}{C_{t+s}} \right] = 0$$

Stationarization

- We want a stationary equilibrium (so that the reference point is a Steady State, not a Balances Growth Path)
- Deflate the model for the growth component Γ_t
- On the example : $x_t = X_t/\Gamma_t$
- Deflated Equilibrium

$$\frac{\theta c_t}{1-h_t} = (1-\alpha)\frac{y_t}{h_t}$$
$$\frac{1}{c_t} = \frac{\beta}{\gamma} E_t\left[\frac{1}{c_{t+1}}\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)\right]$$
$$\gamma k_{t+1} = y_t - c_t + (1-\delta)k_t$$
$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha}$$
$$\lim_{s\longrightarrow\infty} \beta^s \frac{\gamma k_{t+1+s}}{c_{t+s}} = 0$$

- As shown on the previous slide, the solution is generally given by a set of non–linear system of stochastic finite difference equations under rational expectations
- Quite complicated
- In general no analytical solution $\rightsquigarrow$ we need to rely on numerical approximation methods

2. The Standard Real Business Cycle (RBC) Model

To recap

- Very simple RBC model : two first order conditions for firms, two first order onditions for households, and the budget/resource constraint.
- It reduces to three equations, with preferences $U(C) + v(1-h)^{1-\gamma}$:
  1. Euler equation
  $$U'(C_t) = E_t \beta U'(C_{t+1})(1 - \delta + F_k(K_{t+1}, A_{t+1}h_{t+1}))$$
  2. Labour market equilibrium equation
  $$\frac{v(1-h_t)^{-\gamma}}{U'(C_t)} = F_l(K_t, A_t h_t)$$
  3. Resource constraint
  $$Y_t = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t$$
  where
  $$I_t = K_{t+1} - (1-\delta)K_t$$

2. The Standard Real Business Cycle (RBC) Model

To recap

- Many current models of the business cycle can be seen as building off the the RBC model, by introducing wedges. A financial market wedge (μ_{1t}), a labour market wedge (μ_{2t}) and an investment wedge (μ_{3t}).
- Euler equation

$$U'(C_t) = \mu_{1t} E_t \beta U'(C_{t+1}) \left(1 + \delta + F_k(K_{t+1}, A_{t+1} h_{t+1})\right)$$

- Labour market equilibrium equation

$$\frac{v(1-h_t)^{-\gamma}}{U'(C_t)} \mu_{2t} = F_l(K_t, A_t h_t)$$

- Resource constraint

$$Y_t = C_t + I_t = C_t + \mu_{3t}(K_{t+1} - (1-\delta)K_t)$$

- The different theories provide explanation to the wedge, and how is reacts to other shocks.

2. The Standard Real Business Cycle (RBC) Model

To recap

- Very important conceptual breakthrough (closer to pre-Keynesiens). Part of fluctuation may be natural : may need to live with it.
- Conceptually do not need technological regress.
- Issue of optimality can be debated. Positive implications do not change much with efficiency wages, but can change normative issue (constrained optimal ?).
- Very influential piece of thinking (treats macro fluctuations as resulting from an allocation problem, ties in closer with much of the rest economics)
- At closer look, more difficulties : "cleaner" Solow residual much less volatile, co-movement of labour productivity and output almost zero as opposed to being high in model...
- Common noted drawback, conflicts with money non-neutrality (Criticism of real-nominal dichotomy). But how do we know what monetary changes do ?

- A linear model is easy to solve
- A typical way of solving a DSGE is to linearize it
- Given non-linear system of form

$$g(z_{t+1}) = h(z_t)$$

- Define steady state as z
- rewrite system as

$$\ln g(e^{\ln z_{t+1}}) = \ln h(e^{\ln z_t})$$

Take first order Taylor's series approximation on both sides, and equate.

$$\left(\frac{zg'(z)}{g(z)}\right)(\ln z_{t+1} - \ln z) = \left(\frac{zh'(z)}{h(z)}\right)(\ln z_t - lnz)$$

- We will then solve a system of linear Rational Expectations difference equation model

- Economic agents have to make a decision based on their forecast of some future variable.
- In the most general case, the entire probability distribution of that variable will affect their optimal choice.
- The rational expectations hypothesis states that people take into account the *actual* distribution of the relevant variable they want to forecast.
- If expectations are rational, we have to solve simultaneously for the equilibrium distribution and for the rest of the model
- We restrict to linear models in which the distribution of endogenous variables of interest only matters through its first moment, i.e. its mathematical expectation.

- Agricultural market where producers have to decide how much to produce prior to knowing the equilibrium price

$$\begin{aligned} y_t^d &= a - bp_t + \varepsilon_{dt} \\ y_t^s &= c + dp_t^a + \varepsilon_{st}, \\ y_t^s &= y_t^d. \end{aligned}$$

- Here, p =price,ε_d=demand shock, ε_s=supply shock, y^s =supply, y^d =demand,p^a = anticipated price.

- The Rational Expectations Hypothesis assumes that

$$p_t^a = E(p_t \mid I_t),$$

  where E = mathematical expectation operator and I_t = information set of the price-setters when they set y_t^s.
- Typically, $I_t = \{$past values of all the variables, the model, how to solve it$\}$
- If farmers set y_t^s at date $t-1$, :

$$p_t^a = E_{t-1} p_t.$$

- General principle :
  - × take expectations in all the equations and then solve for the expected variables,
  - × then substitute this in all the equations where expectations enter,
  - × this allows to solve for the actual equilibrium variables.
- Let us try this here.

$$\begin{aligned} y_t^d &= a - bp_t + \varepsilon_{dt} \\ y_t^s &= c + dp_t^a + \varepsilon_{st}, \\ y_t^s &= y_t^d. \end{aligned}$$

- The model can be collapsed to

$$a - bp_t + \varepsilon_{dt} = c + dE_{t-1}p_t + \varepsilon_{st}. \tag{1}$$

- Take expectations :

$$a - bE_{t-1}p_t + E_{t-1}\varepsilon_{dt} = c + dE_{t-1}p_t + E_{t-1}\varepsilon_{st}.$$

- Thus

$$E_{t-1}p_t = \frac{a - c + E_{t-1}(\varepsilon_{dt} - \varepsilon_{st})}{b + d}.$$

- substitute this into (1) to solve for p_t :

$$p_t = \frac{1}{b}(\varepsilon_d - \varepsilon_s) - \frac{d}{b(b+d)} E_{t-1}(\varepsilon_d - \varepsilon_s) + \frac{a-c}{b+d}. \tag{2}$$

- This is the solution.

- We can use another method called the method of undetermined coefficients
- This is akin to the notion of Markov perfect equilibrium in game theory.
- Method (in this particular case) :
  - Identify the relevant state space
  - Look for a solution as a (linear) function of this state vector
  - Given such a function, I can compute all the expectations I need
  - Subsitute this function and the appropriate formula for expectations in all places where it is needed
  - Identify the coefficients on each state variable
  - This gives us equations in the coefficients that allow us to compute them

- Assume $E_{t-1}\varepsilon_{dt} = E_{t-1}\varepsilon_{st} = 0$. *(to simplify the algebra, but this is not needed)*
- The relevant state space is $(\varepsilon_d, \varepsilon_s)$.
- Look for a solution of the form :

$$p_t = \alpha\varepsilon_{dt} + \beta\varepsilon_{st} + \gamma.$$

- Under the assumed solution, we have

$$E_{t-1}p_t = \gamma.$$

- Substituting into (1) we get

$$a - b\left(\alpha\varepsilon_{dt} + \beta\varepsilon_{st} + \gamma\right) + \varepsilon_{dt} = c + d\gamma + \varepsilon_{st}.$$

- This must hold for all realizations of ε_d and ε_s and therefore we must have

$$\begin{aligned} a - b\gamma &= c + d\gamma \\ -\alpha b + 1 &= 0 \\ -\beta b &= 1. \end{aligned}$$

- Therefore

$$\begin{aligned} \gamma &= \frac{a-c}{b+d} \\ \alpha &= 1/b \\ \beta &= -1/b. \end{aligned}$$

- Hence the solution

$$p_t = \frac{1}{b}(\varepsilon_d - \varepsilon_s) + \frac{a-c}{b+d},$$

- which is equivalent to (2) when $E_{t-1}\varepsilon_{dt} = E_{t-1}\varepsilon_{st} = 0$ is assumed.

- Suppose now the supply curve is

$$y_t^s = c + dE_t p_{t+1} + \varepsilon_{st}$$

- Eliminating y between supply and demand :

$$p_t = \frac{a-c}{b} + \frac{-d}{b} E_t p_{t+1} + \frac{\varepsilon_{dt} - \varepsilon_{st}}{b}$$

- and assuming for simplicity that the constant terms are zero, we end up with

$$p_t = \rho E_t p_{t+1} + \eta_t, \tag{3}$$

where η_t is a shock.

Solution by forward integration

- (3) implies $p_{t+i} = \rho E_{t+i} p_{t+i+1} + \eta_{t+i} \quad \forall i$
- Taking expectations

$$E_t p_{t+i} = \rho E_t p_{t+i+1} + E_t \eta_{t+i}$$

- Substitute forward (*forward integration*) :

$$p_t = \rho^k E_t p_{t+k} + \sum_{j=0}^{k-1} \rho^j E_t \eta_{t+j} \tag{4}$$

- Assume stationary shocks and impose $E_t p_{t+k}$ is bounded $\forall k$.
- (stochastic equivalent of selecting the non-explosive saddle-path as the unique solution.)
- Then if $\mid \rho \mid < 1$, then $\lim_{k\to\infty} \rho^k E_t p_{t+k} = 0$, hence

$$p_t = \sum_{j=0}^{\infty} \rho^j E_t \eta_{t+j}.$$

$$p_t = \sum_{j=0}^{\infty} \rho^j E_t \eta_{t+j}.$$

- With $| \rho | < 1$, this is the unique stationary solution to the rational expectations equation (3).

$$p_t = \sum_{j=0}^{\infty} \rho^j E_t \eta_{t+j}.$$

- Suppose $\mid \rho \mid > 1$, then the solution cannot be unique.
- Why ?
- For any solution p_t, $p_t + x_t$ is also a solution under some conditions on x
- x_t is a random variable (sunspot) satisfying

$$x_{t+1} = \frac{1}{\rho} x_t + u_{t+1}, \tag{5}$$

  with u any shock such that $E_t u_{t+1} = 0$.
- Furthermore, we can construct at least one non explosive solution by picking $p_{t+1} = \frac{1}{\rho}(p_t - \eta_t)$.
- Hence in this case we have a continuum of equilibria.

Undetermined coefficients

- We can apply the method of undetermined coefficients to this problem if we know the distribution of η.
- Assume η follows an AR(1) :

$$\eta_t = \alpha\eta_{t-1} + \omega_t,$$

with ω_t iid and with zero mean.

- The only relevant state variables are η_{t-1} and ω_t so we look for a solution of the form

$$p_t = \lambda\eta_{t-1} + \beta\omega_t.$$

- Then

$$\begin{aligned} E_t p_{t+1} &= E_t(\lambda\eta_t + \beta\omega_{t+1}) \\ &= \lambda\eta_t \\ &= \lambda(\alpha\eta_{t-1} + \omega_t). \end{aligned}$$

Undetermined coefficients (continued)

- We can then rewrite (3) as

$$\lambda\eta_{t-1} + \beta\omega_t = \rho\lambda(\alpha\eta_{t-1} + \omega_t) + \alpha\eta_{t-1} + \omega_t.$$

- Identifying, we get

$$\begin{aligned} \lambda &= \rho\lambda\alpha + \alpha \\ \beta &= \rho\lambda + 1 \end{aligned}$$

- Hence

$$\begin{aligned} \lambda &= \frac{\alpha}{1 - \rho\alpha}; \\ \beta &= \frac{1}{1 - \rho\alpha}. \end{aligned}$$

- We can check that if $\mid \rho \mid < 1$, this is indeed correct :

$$\begin{aligned}\sum_{j=0}^{\infty} \rho^j E_t \eta_{t+j}. &= \sum_{j=0}^{\infty} \rho^j \alpha^j \eta_t \\ &= \frac{1}{1-\rho\alpha}(\alpha\eta_{t-1} + \omega_t) \\ &= \lambda\eta_{t-1} + \beta\omega_t\end{aligned}$$

- If $\mid \rho \mid > 1$, this is still a solution, and it is the unique "linear Markov" solution, i.e. the only one which is a linear combination of the relevant state space.
- But many other solutions can be constructed by adding a sunspot x satisfying (5).
- Note : when we use undetermined coefficient, we do not check for determinacy.

- General linear model (Blanchard-Kahn)

$$\begin{pmatrix} K_t \\ X_t \end{pmatrix} = A \begin{pmatrix} K_{t-1} \\ E_t X_{t+1} \end{pmatrix} + U_t, \tag{6}$$

- K_t is an $m \times 1$ vector of state (*predetermined*) variables, meaning that K_{t-1} is fixed at the beginning of period t,
- X_t is an $n \times 1$ vector of non-predetermined variables, i.e. expected future values of those variables, rather than lagged ones, affect current outcomes.
- A is an $(m+n) \times (m+n)$ matrix
- U an $(m+n) \times 1$ vector of shocks.

- Let

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \quad , \quad U_t = \begin{pmatrix} U_{0t} \\ U_{1t} \end{pmatrix},$$

- then

$$\begin{aligned} K_t &= A_{00}K_{t-1} + A_{01}E_tX_{t+1} + U_{0t} \\ X_t &= A_{10}K_{t-1} + A_{11}E_tX_{t+1} + U_{1t}. \end{aligned}$$

- We can solve for E_tX_{t+1} and write the whole system with backward-looking dynamics :

$$\begin{aligned} \begin{pmatrix} K_t \\ E_tX_{t+1} \end{pmatrix} &= \begin{pmatrix} A_{00} - A_{01}A_{11}^{-1}A_{10} & A_{01}A_{11}^{-1} \\ -A_{11}^{-1}A_{10} & A_{11}^{-1} \end{pmatrix} \begin{pmatrix} K_{t-1} \\ X_t \end{pmatrix} \\ &+ \begin{pmatrix} U_{0t} - A_{01}A_{11}^{-1}U_{1t} \\ -A_{11}^{-1}U_{1t} \end{pmatrix} \\ &= M \begin{pmatrix} K_{t-1} \\ X_t \end{pmatrix} + V_t. \end{aligned} \tag{7}$$

$$\begin{pmatrix} K_t \\ E_t X_{t+1} \end{pmatrix} = M \begin{pmatrix} K_{t-1} \\ X_t \end{pmatrix} + V_t. \tag{7}$$

- Let us start with the simple case where M is diagonal, say $M = \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix}$, where $D_0 = \begin{pmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & ... \\ ... & ... & ... & 0 \\ 0 & ... & 0 & d_m \end{pmatrix}$ and $D_1 = \begin{pmatrix} d'_1 & 0 & ... & 0 \\ 0 & d'_2 & ... & ... \\ ... & ... & ... & 0 \\ 0 & ... & 0 & d'_n \end{pmatrix}$ and $V_t = \begin{pmatrix} V_{0t} \\ V_{1t} \end{pmatrix}$
-

- Then, iterating (7),

$$K_{t+i} = D_0^{i+1} K_{t-1} + \sum_{j=0}^{i} D_0^{i-j} V_{0t+j}.$$

- For K_{t+i} to remain bounded, we need

$$\mid d_i \mid < 1, i = 1, ..., m. \tag{8}$$

- Similarly

$$\begin{aligned} E_t X_{t+i} &= D_1^i X_t + \sum_{j=0}^{i-1} D_1^{i-1-j} E_t V_{1t+j} \\ &= D_1^i \left(X_t + \sum_{j=0}^{i-1} D_1^{-1-j} E_t V_{1t+j} \right). \end{aligned}$$

- Consider first a k such that $\mid d'_k \mid < 1$. Let x_{kt} (resp. v_{kt}) be the k-th term in X_t (resp. V_{1t}).
- Then

$$\begin{aligned} E_t x_{t+i} &= (d'_k)^i x_t + \sum_{j=0}^{i-1} (d'_k)^{i-1-j} E_t v_{k,t+j} \\ &\iff (d'^{-1}_k)^i E_t x_{t+i} - \sum_{j=0}^{i-1} (d'^{-1}_k)^{1+j} E_t v_{k,t+j} = x_t. \end{aligned}$$

- This is similar to (4) and since $\mid d'^{-1}_k \mid > 1$ there is no unique initial value of x_t yielding a non-explosive path for subsequent expectations.

- Second, assume conversely that

$$| d_k' | > 1 \text{ for } k = 1, ..., n. \tag{9}$$

- Then all the terms in D_1^i go to infinity as $i \to \infty$.
- Thus it must be that $\lim_{i \to +\infty} X_t + \sum_{j=0}^{i-1} D_1^{-1-j} E_t V_{1t+j} = 0$. Therefore,

$$X_t = -\sum_{j=0}^{\infty} \left(D_1^{-1}\right)^{1+j} E_t V_{1t+j}.$$

$$| d_i | < 1, i = 1, ..., m. \tag{8}$$

$$| d'_k | > 1 \text{ for } k = 1, ..., n. \tag{9}$$

- To summarize
  - × To have a unique non-explosive solution, (8) and (9) must hold
  - × If (8) is violated, all solutions are explosive
  - × If (9) is violated, there is a continuum of non-explosive solutions.

3.3. A general model

The non diagonal case

- Assume now that the matrix M is not diagonal.
- The idea is to transform the model into a diagonal one.
- Generically, M is diagonalizable in the complex plane $\mathbb{C}$.
- Thus there exists P invertible such that

$$M = P^{-1}DP,$$

with D diagonal :

$$D = \begin{pmatrix} d_0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & d_m & 0 & \dots & \dots & \dots \\ \dots & \dots & 0 & d_{m+1} & 0 & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & d_{m+n} \end{pmatrix} = \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix}$$

- The diagonal coefficients are the eigenvalues of M.
- Eigenvalues are ranked by ascending module :
  $\mid d_0 \mid \leq \mid d_1 \mid \leq ... \leq \mid d_m \mid \leq \mid d'_1 \mid \leq \mid d'_2 \mid \leq ... \leq \mid d'_n \mid$.
- (7) rewrite

$$P \begin{pmatrix} K_t \\ E_t X_{t+1} \end{pmatrix} = D \left(P \begin{pmatrix} K_t \\ X_t \end{pmatrix} \right) + PV_t.$$

- Let $Z_t = P \begin{pmatrix} K_{t-1} \\ X_t \end{pmatrix}$;
- We are back to the diagonal case with this change of variable.

3.4. How general is the general model ?

More lags for K

- Suppose that we have terms in K_{t-k}, $k > 1$.
- The solution is to add the variables $(K_{t-1}, ..., K_{t-k+1})$ to the state vector.
- Suppose we have this equation $K_t = aK_{t-1} + bK_{t-k}$
- Let us assume to save on notation that K is a scalar.
- Then the vector $\begin{pmatrix} K_t \\ K_{t-1} \\ ... \\ K_{t-k+1} \end{pmatrix}$ satisfies

$$\begin{pmatrix} K_t \\ K_{t-1} \\ ... \\ K_{t-k+1} \end{pmatrix} = \begin{pmatrix} a & 0 & ... & b \\ 1 & 0 & ... & ... \\ 0 & ... & ... & ... \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} K_{t-1} \\ K_{t-2} \\ ... \\ K_{t-k} \end{pmatrix},$$

which only involves the first lag of this vector.

3.4. How general is the general model ?

More lags for X

- Suppose we have terms in $E_t X_{t+k}$, $k > 1$.
- Let's add $E_t X_{t+1}, ..., E_t X_{t+k-1}$ to the jump variables
- Suppose that we have

$$X_t = aE_t X_{t+1} + bE_t X_{t+k}.$$

- Then the vector $\Psi_t = (X_t, E_t X_{t+1}, ..., E_t X_{t+k-1})'$ satisfies

$$\begin{aligned}\Psi_t &= \begin{pmatrix} X_t \\ E_t X_{t+1} \\ ... \\ E_t X_{t+k-1} \end{pmatrix} = \begin{pmatrix} a & 0 & ... & b \\ 1 & 0 & ... & ... \\ 0 & ... & ... & ... \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_t X_{t+1} \\ E_t X_{t+2} \\ ... \\ E_t X_{t+k} \end{pmatrix} \\ &= \begin{pmatrix} a & 0 & ... & b \\ 1 & 0 & ... & ... \\ 0 & ... & ... & ... \\ 0 & 0 & 1 & 0 \end{pmatrix} E_t \Psi_{t+1}.\end{aligned}$$

-

- Assume we have a system of n linear difference equations in the following form

$$\mathbb{E}_t Y_{t+1} = \mathrm{A}\, Y_t + \mathrm{B}\, V_t \tag{10}$$

- where $\mathbb{E}_t Y_{t+1}$ represents the expectation of Y_{t+1} given the information set available at time t, V_t is a $(k \times 1)$ vector of exogeneous driving processes and Y_t is a $(n \times 1)$ vector where the first ℓ variables are state variables and the remaining $n - \ell$ are jump/free variables.

- We now consider the decomposition

$$P^{-1}\mathrm{A}P = \Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} \tag{11}$$

- where Λ is a diagonal matrix with the eigenvalues of A as elements in increasing size.
- In the regular there will be ℓ elements of Λ that will be smaller than 1 and $n - \ell$ elements that will be greater than 1.

- Now consider the following transformation

$$\tilde{Y}_t = P^{-1} Y_t$$

with

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$

- P_{11} is an $\ell \times \ell$ matrix, P_{12} is an $\ell \times (n-\ell)$ matrix, P_{21} is an $(n-\ell) \times \ell$ matrix and P_{22} is an $(n-\ell) \times (n-\ell)$ matrix.

- Now, premultiplying (10) by P^{-1}, we obtain

$$P^{-1}Y_{t+1} = P^{-1}\mathrm{A}\ \underbrace{PP^{-1}}_{\mathrm{I}}\ Y_t + P^{-1}\mathrm{B}V_t$$

- so that

$$\tilde{Y}_{t+1} = \Lambda\tilde{Y}_t + \tilde{\mathrm{B}}V_t \tag{12}$$

- that is

$$\begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1,t} \\ \tilde{Y}_{2,t} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} V_t$$

- with $\tilde{B}_1$ an $\ell \times k$ matrix and $\tilde{B}_2$ and $(n - \ell) \times k$ matrix.

- We can rewrite system as

$$(I - \Lambda L)\tilde{Y}_{t+1} = \tilde{\mathrm{B}} V_t$$

- or

$$\begin{bmatrix} I - \Lambda_{11} L & 0 \\ 0 & I - \Lambda_{22} L \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} V_t$$

- Since all the eigenvalues in Λ_{22} are greater than one, we can solve for $\tilde{Y}_2$ using a forward expansion.

$$\tilde{Y}_{2,t+1} = -\Lambda_{22}^{-1} L^{-1} (I - \Lambda_{22}^{-1} L^{-1})^{-1} \tilde{B}_2 V_t \tag{13}$$

- The last expression is the matrix equivalent of

$$-\frac{\lambda_2^{-1} L^{-1}}{1 - \lambda_2^{-1} L^{-1}}$$

- when $\lambda_2 > 1$.

Solving system of difference equations

- From (11), we know that
$$P\tilde{Y}_t = Y_t$$
- so that
$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1,t} \\ \tilde{Y}_{2,t} \end{bmatrix} = \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix}$$
- and in particular,
$$P_{11}\tilde{Y}_{1,t} + P_{12}\tilde{Y}_{2,t} = Y_{1,t}$$

- Multiplying the latter expression by P_{11}^{-1}, we get

$$\tilde{Y}_{1,t} = P_{11}^{-1} Y_{1,t} - P_{11}^{-1} P_{12} \tilde{Y}_{2,t}$$

- moving forward the latter expression, we obtain

$$\tilde{Y}_{1,t+1} = P_{11}^{-1} Y_{1,t+1} - P_{11}^{-1} P_{12} \tilde{Y}_{2,t+1}$$

- Since (12) says that

$$\tilde{Y}_{1,t+1} = \Lambda_{11}\tilde{Y}_{1,t} + \tilde{B}_1 V_t$$

- we have

$$P_{11}^{-1}Y_{1,t+1} - P_{11}^{-1}P_{12}\tilde{Y}_{2,t+1} = \Lambda_{11}\left(P_{11}^{-1}Y_{1,t} - P_{11}^{-1}P_{12}\tilde{Y}_{2,t}\right) + \tilde{B}_1 V_t$$

- Premultiplying the previous expression by P_{11} and with a bit of algebra, we obtain

$$Y_{1,t+1} = P_{11}\Lambda_{11}P_{11}^{-1}Y_{1,t} - P_{11}\Lambda_{11}P_{11}^{-1}P_{12}\tilde{Y}_{2,t} + P_{12}\tilde{Y}_{2,t+1} + \tilde{B}_1 V_t$$

- We can use (13) to substitute for $\tilde{Y}_{2,t+1}$ and $\tilde{Y}_{2,t}$ to find the solution to our state variables.
- We can see how states depend on past states and expectations of all future variables
- Using $Y_{2,t} = P_{21}\tilde{Y}_{1,t} + P_{22}\tilde{Y}_{22,t}$, we then get solution for jump variables
- Given a law of motion for V_t, we get the state space representation.

- Household problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \cdot u(c_t, \ell_t^s) \tag{14}$$

- subject to

$$c_t + a_{t+1} = w_t \ell_t^s + (1 + r_t) \cdot a_t + \pi_t \tag{15}$$

- Writing the Lagrangian

$$L = \max E_0 \sum_{t=0}^{\infty} \beta^t \cdot [u(c_t, \ell_t^s) + \lambda_t [w_t \ell_t^s + (1 + r_t) \cdot a_t + \pi_t - c_t - a_{t+1}]] \tag{16}$$

- first-order conditions

$$\frac{\partial L}{\partial c_t} : u_c(c_t, \ell_t^s) = \lambda_t \qquad t = 0, 1, 2, ... \tag{17}$$

$$\frac{\partial L}{\partial \ell_t} : u_\ell(c_t, \ell_t^s) = w_t \lambda_t \qquad t = 0, 1, 2, ... \tag{18}$$

$$\frac{\partial L}{\partial a_{t+1}} : \lambda_t = \beta E_t \left(\lambda_{t+1} \cdot (1 + r_{t+1})\right) \qquad t = 0, 1, 2, ... \tag{19}$$

- Combining (17) and (18), we obtain

$$\frac{u_\ell(c_t, \ell_t^s)}{u_c(c_t, \ell_t^s)} = w_t \qquad t = 0, 1, 2, ... \tag{20}$$

- Combining (17) and (19) leads to

$$u_c(c_t, \ell_t^s) = \beta E_t \left(u_c(c_{t+1}, \ell_{t+1}^s) \cdot (1 + r_{t+1}) \right) \qquad t = 0, 1, 2, ... \tag{21}$$

3.6. Example with RBC model

- Firm's problem

$$\max_{K_t, L_t} F(K_t, A_t L_t) - w_t L_t^d - R_t K_t \tag{22}$$

- This gives us the usual first-order conditions

$$L_t^d : A_t F_L(K_t, A_t L_t^d) = w_t \tag{23}$$

$$K_t : F_K(K_t, A_t L_t^d) = R_t \tag{24}$$

3.6. Example with RBC model

- Competitive equilibrium
- In this economy, a *competitive equilibrium* is a set of prices $\{w_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$, $\{R_t\}_{t=0}^{\infty}$ and a set of sequences $\{c_t\}_{t=0}^{\infty}$, $\{\ell_t\}_{t=0}^{\infty}$, $\{a_t\}_{t=0}^{\infty}$, $\{k_t\}_{t=0}^{\infty}$ such that
  1. given $\{w_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$, the quantities solve the household's maximization problem 14;
  2. given $\{w_t\}_{t=0}^{\infty}$, $\{R_t\}_{t=0}^{\infty}$, the quantities solve the firm's maximization problem 22;
  3. all market clears.

- Equilibrium conditions

$$A_t F_L(K_t, A_t N_t \ell_t) = \frac{u_\ell(c_t, \ell_t)}{u_c(c_t, \ell_t)} \tag{25}$$

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, A_t \ell_t) \tag{26}$$

$$u_c(c_t, \ell_t) = \beta E_t \left(u_c(c_{t+1}, \ell_{t+1}) \cdot (1 + F_K(K_{t+1}, A_{t+1} N_{t+1} \ell_{t+1}) - \delta) \right) \tag{27}$$

3.6. Example with RBC model

- From our equilibrium conditions, we can then find a solution. This solution can take one of the following three representations :
  1. The state-space representation : $K_{t+1} = g(K_t, A_t)$;
  2. The impulse-response function : $K_{t+1} = \nu(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...)$
  3. The moments of the variables, mainly their variances and covariances

3.6. Example with RBC model

- Since we know how to analyse linear and stationnary models, our strategy will be to
  1. render our system stationary (by taking ratios or detrending) and then ;
  2. take a linear approximation around the steady state.

3.6. Example with RBC model

- If A_t is stationary around a stochastic trend, for example, if

$$A_t = A_0 g(t) V_t$$

- where V_t is a log normal stationary process, then we can render our system stationary by detrending all the relevant variables by A's trend.
- Therefore, we define

$$\tilde{c}_t = \frac{c_t}{A_0 g(t)} \qquad \text{and} \qquad \tilde{k}_{t+1} = \frac{k_{t+1}}{A_0 g(t)}$$

- which are the variables of the system in deviation from trend. Note that that our original system can be written this way with only slight modifications (discuss example with trend growth). Hence we want to express our system in $\hat{c}_t, \hat{k}_{t+1}$, where

$$\hat{c}_t = \frac{\tilde{c}_t - \tilde{c}^*}{\tilde{c}^*} \qquad \text{and} \qquad \hat{k}_{t+1} = \frac{\tilde{k}_{t+1} - \tilde{k}^*}{\tilde{k}^*}$$

- Finding an approximation to the one-sector neo-classical model
- In the case of fixed labour supply, the equilibrium conditions for the one-sector model are

$$u_c(c_t) = \beta E_t \left(u_c(c_{t+1}) \cdot (1 - \delta + F_k(k_{t+1}, A_{t+1})) \right) \tag{28}$$

$$c_t + k_{t+1} - (1 - \delta)k_t = F(k_t, A_t) \tag{29}$$

- Taking a first-order linear approximation around the steady state gives us

$$\begin{aligned} u_{cc}(\bar{c}) \cdot (c_t - \bar{c}) &= \beta(1 - \delta + F_k(\bar{k}, \bar{A}))u_{cc}(\bar{c})E_t(c_{t+1} - \bar{c}) \\ &\quad + \beta u_c(\bar{c})F_{kk}(\bar{k}, \bar{A})E_t(k_{t+1} - \bar{k}) \\ &\quad + \beta u_c(\bar{c})F_{kA}(\bar{k}, \bar{A})E_t(A_{t+1} - \bar{A}) \end{aligned} \tag{30}$$

- Now, using the fact that at the steady state,

$$u_c(\bar{c}) = \beta \left(u_c(\bar{c}) \cdot (1 - \delta + F_k(\bar{k}, \bar{A})) \right)$$
$$1 = \beta(1 - \delta + F_k(\bar{k}, \bar{A}))$$

- and also

$$u_c(\bar{c}) = \beta(1 - \delta) \cdot u_c(\bar{c}) + \beta u_c(\bar{c}) \cdot F_k(\bar{k}, \bar{A})$$
$$u_c(\bar{c})(1 - \beta(1 - \delta)) = \beta u_c(\bar{c}) \cdot F_k(\bar{k}, \bar{A})$$
$$\frac{(1 - \beta(1 - \delta))}{F_k(\bar{k}, \bar{A})} = \beta$$

3.6. Example with RBC model

- we can rewrite (30) as

$$\begin{aligned}\frac{\bar{c}\cdot u_{cc}(\bar{c})}{u_c(\bar{c})}\cdot\frac{(c_t-\bar{c})}{\bar{c}} &= \frac{\bar{c}\cdot u_{cc}(\bar{c})}{u_c(\bar{c})}E_t(\frac{c_{t+1}-\bar{c}}{\bar{c}}) \\ &+\beta\bar{k}F_{kk}(\bar{k},\bar{A})E_t(\frac{k_{t+1}-\bar{k}}{\bar{k}}) \\ &+\beta\bar{A}F_{kA}(\bar{k},\bar{A})E_t(\frac{A_{t+1}-\bar{A}}{\bar{A}})\end{aligned}$$

- and finally,

$$\begin{aligned}\frac{\bar{c}\cdot u_{cc}(\bar{c})}{u_c(\bar{c})}\cdot\frac{(c_t-\bar{c})}{\bar{c}} &= \frac{\bar{c}\cdot u_{cc}(\bar{c})}{u_c(\bar{c})}E_t(\frac{c_{t+1}-\bar{c}}{\bar{c}}) \\ &+(1-\beta(1-\delta))\frac{\bar{k}F_{kk}(\bar{k},\bar{A})}{F_k(\bar{k},\bar{A})}E_t(\frac{k_{t+1}-\bar{k}}{\bar{k}}) \\ &+(1-\beta(1-\delta))\frac{\bar{A}F_{kA}(\bar{k},\bar{A})}{F_k(\bar{k},\bar{A})}E_t(\frac{A_{t+1}-\bar{A}}{\bar{A}})\end{aligned} \tag{31}$$

- For condition (29), we have

$$c_t + k_{t+1} = F(k_t, A_t) + (1-\delta)k_t$$

$$F_k(\bar{k},\bar{A})(k_t-\bar{k}) + F_A(\bar{k},\bar{A})(A_t-\bar{A}) = (c_t-\bar{c}) + (k_{t+1}-\bar{k}) - (1-\delta)(k_t-\bar{k})$$

$$F_k(\bar{k},\bar{A})\bar{k}\frac{(k_t-\bar{k})}{\bar{k}} + \bar{A}F_A(\bar{k},\bar{A})\frac{(A_t-\bar{A})}{\bar{A}} = \bar{c}\frac{(c_t-\bar{c})}{\bar{c}} + \bar{k}\frac{(k_{t+1}-\bar{k})}{\bar{k}} - (1-\delta)\bar{k}\frac{(k_t-\bar{k})}{\bar{k}}$$

$$\frac{F_k(\bar{k},\bar{A})\bar{k}}{F(\bar{k},\bar{A})}\frac{(k_t-\bar{k})}{\bar{k}} + \frac{\bar{A}F_A(\bar{k},\bar{A})}{F(\bar{k},\bar{A})}\frac{(A_t-\bar{A})}{\bar{A}} = \frac{\bar{c}}{\bar{y}}\frac{(c_t-\bar{c})}{\bar{c}} + \frac{\bar{k}}{\bar{y}}\frac{(k_{t+1}-\bar{k})}{\bar{k}} - \frac{(1-\delta)\bar{k}}{\bar{y}}\frac{(k_t-\bar{k})}{\bar{k}}$$

3.6. Example with RBC model

- To complete our equation, we must find an expression for $\frac{\bar{c}}{\bar{y}}$ and $\frac{\bar{k}}{\bar{y}}$. In the former case, we have

$$\bar{c} + \bar{k} - (1-\delta)\bar{k} = \bar{y}$$

$$\bar{c} = F(\bar{k}, \bar{A}) - \delta\bar{k} \qquad \Rightarrow \qquad \frac{\bar{c}}{\bar{y}} = 1 - \delta\frac{\bar{k}}{\bar{y}}$$

- while for the latter, we have

$$u_c(\bar{c}) = \beta \cdot u_c(\bar{c}) \cdot (1 - \delta + F_k(\bar{k}, \bar{A}))$$

$$\left(\frac{1}{\beta} - 1 + \delta\right) = F_k(\bar{k}, \bar{A})$$

$$\frac{\bar{k}}{\bar{y}}\left(\frac{1}{\beta} - 1 + \delta\right) = \frac{\bar{k}}{\bar{y}}F_k(\bar{k}, \bar{A})$$

$$\frac{\bar{k}}{\bar{y}} = s \cdot \left(\frac{1}{\beta} - 1 + \delta\right)^{-1}$$

with s representing the capital share of income.

- The solution to the log–linearize version of the model takes the form

$$\begin{aligned} X_{t+1} &= M_x X_t + M_z Z_t \\ Z_{t+1} &= \rho Z_t + \varepsilon_{t+1} \\ Y_t &= P_x X_t + P_z Z_t \end{aligned}$$

  where X_t, Z_t and Y_t collect, respectively, the state variables, the shocks and the variables of interest
- Resembles a VAR model
- In the basic RBC model

$$\begin{aligned} X_t &= \{k_t\} \\ Z_t &= \{a_t\} \\ Y_t &= \{y_t, c_t, i_t, h_t\} \end{aligned}$$

4. Quantitative Evaluation of The Standard RBC Model

Parametric Example

- $U(C_t, \ell_t) = \log(C_t) + \theta \log(\ell_t), \qquad Y_t = A_t K_t^{\alpha} (\Gamma_t h_t)^{1-\alpha}$
- $\Gamma_t = \Gamma_0 \gamma^t$
- Equilibrium

$$\frac{\theta C_t}{1 - h_t} = (1 - \alpha) \frac{Y_t}{h_t}$$

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \right) \right]$$

$$K_{t+1} = Y_t - C_t + (1 - \delta) K_t$$

$$Y_t = A_t K_t^{\alpha} (\Gamma_t h_t)^{1-\alpha}$$

$$\lim_{s \longrightarrow \infty} \beta^s E_t \left[\frac{K_{t+1+s}}{C_{t+s}} \right] = 0$$

$$\log(A_t) = \rho \log(A_{t-1}) + (1 - \rho) \log(\overline{A}) + \varepsilon_t$$

$$\varepsilon \rightsquigarrow \mathcal{N}(0, \sigma^2)$$

- Need to assign numerical values to the parameters
- This is the **calibration** step
- What does calibration mean ?
  - × Make explicit use of the model to set the parameters
  - × A lot of discipline, but no systematic recipe
  - × Have to set $(\alpha, \beta, \theta, \delta, \gamma, \rho, \sigma, \overline{A})$
  - × Use data : $(\Delta \log Y, k/y, i/k, h, wh/y)$
  - × Compute $(\Delta \log Y, k/y, i/k, h, wh/y)$ in the model to set $(\alpha, \beta, \theta, \delta, \gamma)$
  - × Then compute A_t on the data and estimate ρ and σ.

4. Quantitative Evaluation of The Standard RBC Model

Quantitative Evaluation - Calibration

- In the data $\Delta y = 0.9\%$ per quarter $\rightsquigarrow \gamma = 1.009$.
- In the data $i/k = 0.076$ on annual data. Use capital accumulation to get annual depreciation.

$$i/k(\text{model}) = \gamma - (1 - \delta) \rightsquigarrow \delta = 0.01$$

- In the data $wh/y = 0.6$, in the model $wh/y = 1 - \alpha \rightsquigarrow \alpha = 0.4$.
- In the data $k/y = 3.32$ on annual data. Using the Euler equation

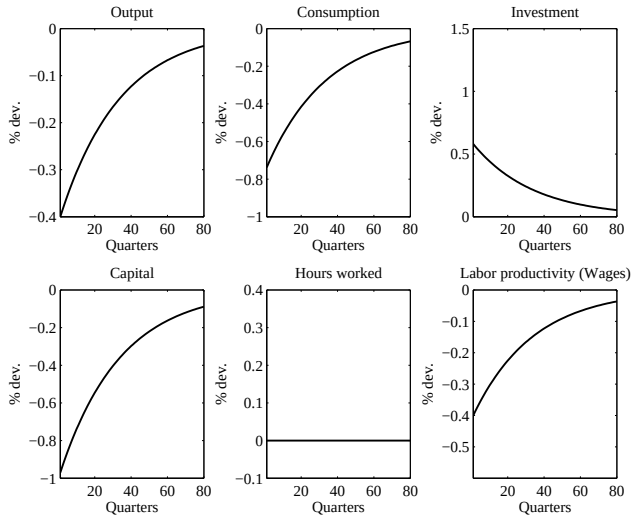
$$\beta = \frac{\gamma}{\alpha y/k + (1 - \delta)}$$

which gives $\beta = 0.98$

- $h = 0.31$, such that $\theta = \frac{(1-\alpha)(1-h)}{hc/y}$
- We set $\overline{A} = 1$ without loss of generality.
- We have $\log(A_t) = \log(y_t) - \alpha \log(k_t) - (1 - \alpha) \log(h_t)$
- Estimate $\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t$
- We obtain $\rho = 0.95$ and $\sigma = 0.0079$.

- Want to see whether capital dynamics can account for BC.
- Perfect foresights dynamics
- We study a case with fixed labor ($h = \overline{h}$) and a variable labor case ($U(c, 1-h)$)

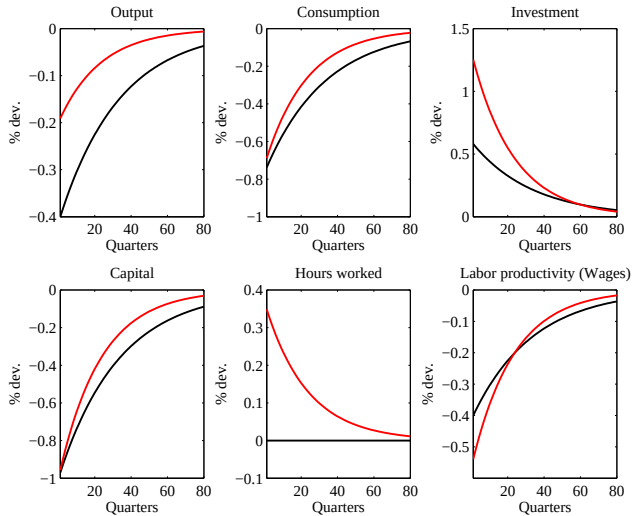
Capital shock - Fixed hours



4. Quantitative Evaluation of The Standard RBC Model

Can the Business Cycle be Driven by Capital Dynamics ?

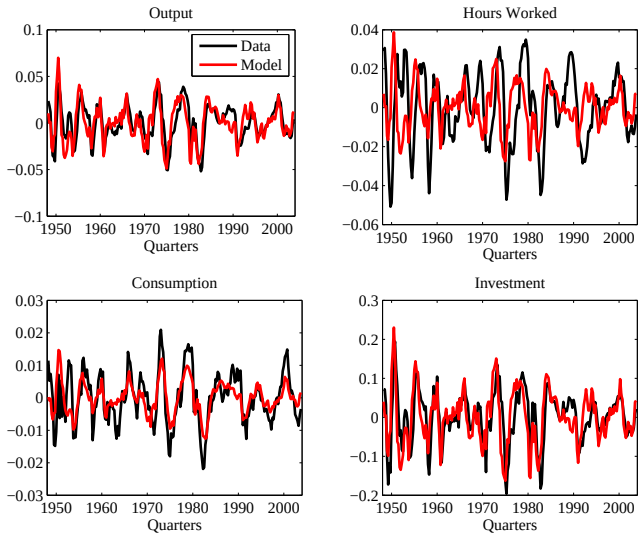
Capital shock - Variable hours



- Capital dynamics cannot be the story for the business cycle.
- More (shocks ?) is needed to understand the BC
- That's why the RBC literature proposes technological shocks (Brock & Mirman, 1972)

- We first stick in the model the estimated ε_t and compare model and data.

A good fit with estimated shocks



- Accounts for the main events in the data
- The model correctly predicts the data : corr(y, y^m)=0.75, corr(c, c^m)=0.73, corr(i, i^m)=0.70.
- BUT : corr(h, h^m)=0.06
- Let's compute unconditional moments in the model (simulate / Hoddrick-Prescott filter / compute moments)

- We already discussed the Hoddrick-Prescott filter
- HP Filter. Idea, decompose $\ln Y_t = g_t + c_t$ by minimizing

$$\sum_{t=1}^{T} c_t^2 = \lambda \sum_{t=3}^{T} [(1-L)^2 g_t]^2$$

  where λ is generally set at 16000 for business cycle analysis
- Implementation often involves apply HP filter to re-trended data from model ($\ln X_t = \hat{x_t} + X_0 + gt$)

U.S. data, 1948-2005

Variable	$\sigma(\cdot)$	$\sigma(\cdot)/\sigma(y)$	$\rho(\cdot, y)$	$\rho(\cdot, h)$	Auto(1)
Output	1.70	–	–	–	0.84
Consumption	0.80	0.47	0.78	–	0.83
Services	1.11	0.66	0.72	–	0.80
Non Durables	0.72	0.42	0.71	–	0.77
Investment	6.49	3.83	0.84	–	0.81
Fixed investment	5.08	3.00	0.80	–	0.88
Durables	5.23	3.09	0.58	–	0.72
Changes in inventories	22.48	13.26	0.48	–	0.40
Hours worked	1.69	1.00	0.86	–	0.89
Labor productivity	0.90	0.53	0.41	0.09	0.69

4. Quantitative Evaluation of The Standard RBC Model

Results from Model Simulations

Variable	$\sigma(\cdot)$	$\sigma(\cdot)/\sigma(y)$	$\rho(\cdot, y)$	$\rho(\cdot, h)$	Auto
Output	1.70	–	–	–	0.84
	1.49	–	–	–	0.68
Consumption	0.80	0.47	0.78	–	0.83
	0.37	0.25	0.81	–	0.82
Investment	6.49	3.83	0.84	–	0.81
	5.00	3.35	0.99	–	0.68
Hours worked	1.69	1.00	0.86	–	0.89
	0.85	0.57	0.98	–	0.68
Labor productivity	0.90	0.53	0.41	0.09	0.69
	0.67	0.45	0.97	0.92	0.72

(model in gray)

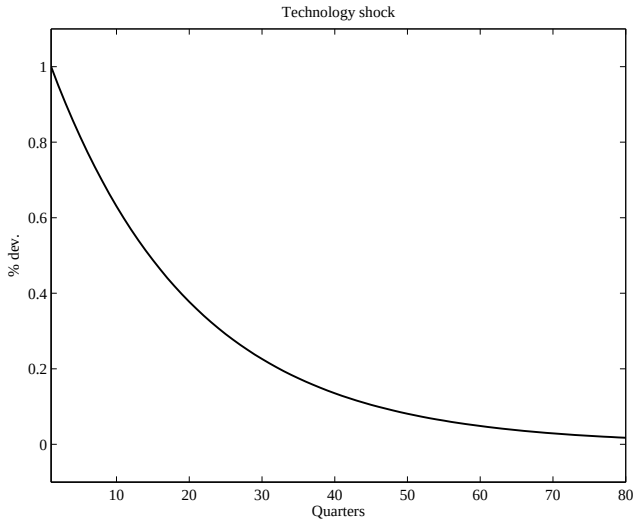
- The model correctly predicts the amplitude, serial correlation and relative variability of fluctuations
- It accounts for a large part of output volatility
- Correct ranking of the volatility of c, i, y, ...
- Large serial correlation, although it is smaller than in the data.
- **But**
- C and N are not volatile enough
- w (and r) are to too procyclical

- The research on RBC became so successful because
  - × Propose a coherent platform to analyse growth and cycles
  - × It somewhat fails such that there is room for work
- Main victory : methodological (part of the toolkit of macroeconomists)
- Main criticisms
  - × incorrect/implausible calibration of parameters (IES, Labor supply) $\rightsquigarrow$ need for sensibility analysis
  - × counterfactual prediction for some prices :
    - real wage is strongly procyclical in the model,
    - CRRA preferences are not compatible with the equity premium
    - price level is too strongly countercyclical

- Key problem : Are technological shocks at the source of BC ?
- Prescott [1986] : Technology shocks account for 70% of output volatility, but
  - × Too volatile
  - × Little evidence of large supply shocks (except oil prices)
  - × Recessions have to be explained by technological regressions
  - × Measurement problems (contamination by demand shocks if increasing returns, imperfect competition, labor hoarding) ; in the growth accounting literature, the SR was a measure of our ignorance, now it is the engine of the model.

- Recall that
$$\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t$$
and ρ is large
- Why do we need so persistent shocks ?
- Because the model possesses weak propagation mechanisms
- Let's see that in details
- See also Cogley and Nason (AER, June 1995)

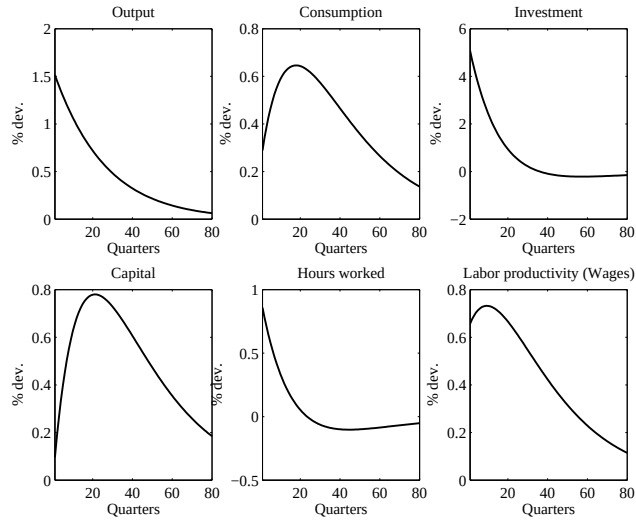
IRF to A Technological Shock



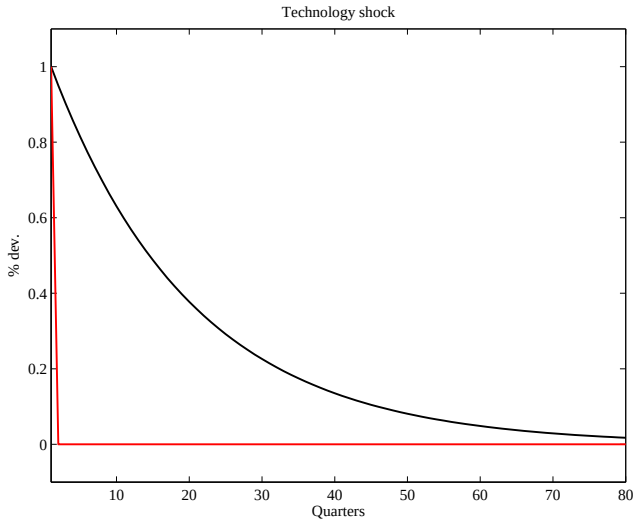
4. Quantitative Evaluation of The Standard RBC Model

Criticisms - The Need for Persistent Shocks

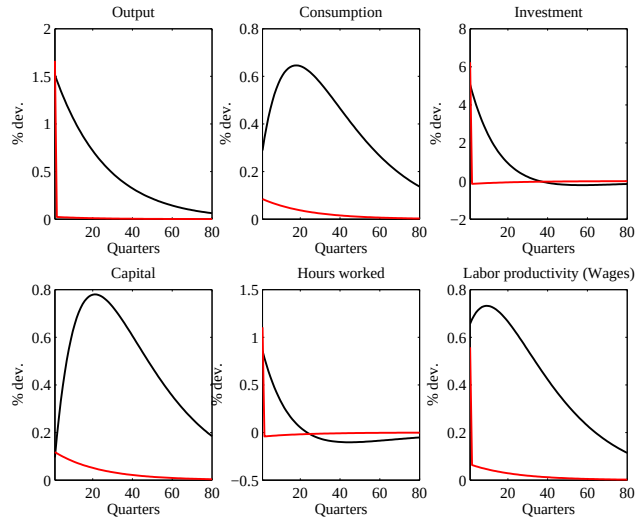
IRF to A Technological Shock



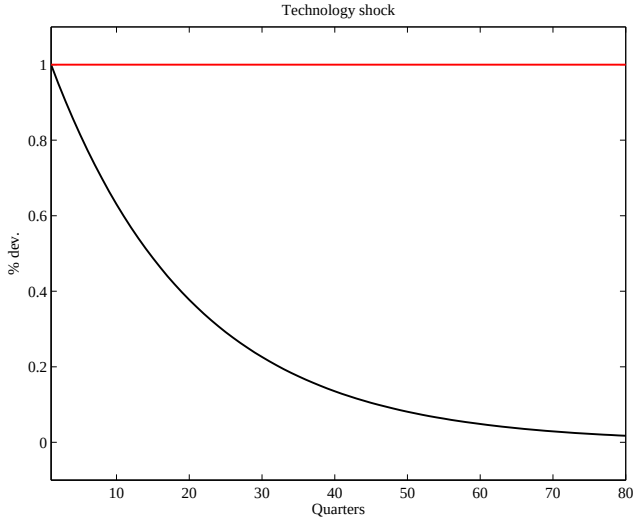
IRF to A Technological Shock : Temporary *vs* Persistent



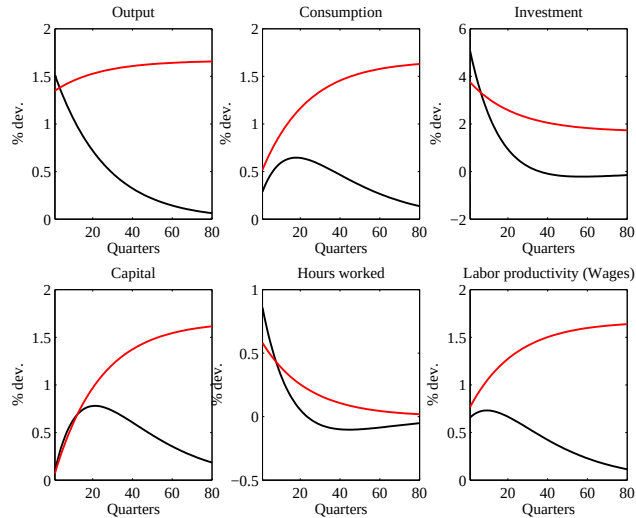
IRF to A Technological Shock : Temporary *vs* Persistent



IRF to A Technological Shock : Persistent *vs* Permanent



IRF to A Technological Shock : Persistent *vs* Permanent



- How to improve the model :
  - × Introducing additional shocks
  - × Improving the propagation mechanisms

- Government Expenditures
- Change the Household's budget constraint :

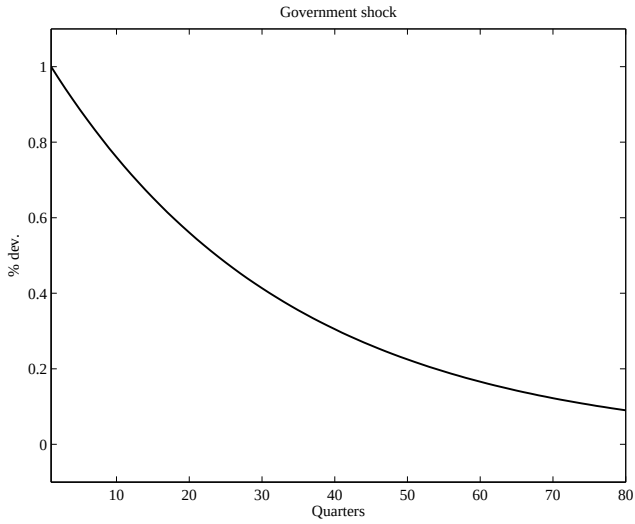
$$B_{t+1} + C_t + I_t + T_t \leqslant (1 + r_{t-1})B_t + W_t h_t + z_t K_t$$

- Government Balanced Budget : $T_t = G_t$
- Government expenditures are modeled as

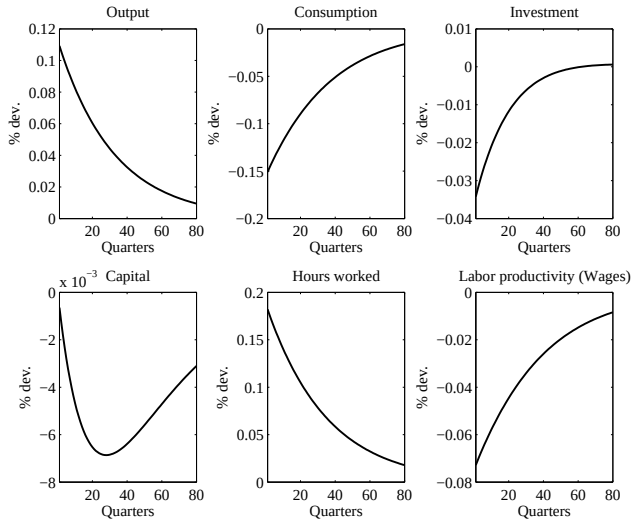
$$\log(G_t) = \rho \log(G_{t-1}) + (1 - \rho_g) \log(\overline{G}) + \varepsilon_{g,t}$$

  with $\varepsilon_{g,t} \rightsquigarrow N(0, \sigma_t^2)$.
- $G/Y = 0.2$, $\rho_g = 0.97$, $\sigma_g = 0.02$.

IRF to A Government Spending Shock



IRF to A Government Spending Shock



Technological and Government Spending Model

Variable	$\sigma(\cdot)$	$\sigma(\cdot)/\sigma(y)$	$\rho(\cdot, y)$	$\rho(\cdot, h)$	
Output	1.70	–	–	–	0.84
	1.43	–	–	–	0.68
Consumption	0.80	0.47	0.78	–	0.83
	0.62	0.43	0.54	–	0.75
Investment	6.49	3.83	0.84	–	0.81
	4.62	3.24	0.97	–	0.68
Hours worked	1.69	1.00	0.86	–	0.89
	0.85	0.59	0.90	–	0.68
Labor productivity	0.90	0.53	0.41	0.09	0.69
	0.76	0.53	0.87	0.58	0.72

(model in gray)

- Rogerson [1984] and Hansen [1985] : work a fixed amount of hours or does not (indivisibility)
- Then, the commodity set is nonconvex. We convexify it by the use of lotteries
- The nice property of this example is that one can define a representative agent, whose preferences are different from the original agents
- Preferences

$$U(C_{it}, 1-h_{it}) = \begin{cases} \log(C^e_{it}) + \theta \log(1-h_0) & \text{if she works} \\ \log(C^u_{it}) + \theta \log(1) & \text{if not} \end{cases}$$

- Randomly drawn to work with probability $\pi_t = \frac{h_t}{h_0}$

$$\begin{aligned} EU_t &= \pi_t \left(\log(C^e_{it}) + \theta \log(1-h_0)\right) + (1-\pi_t)\left(\log(C^u_{it}) + \theta \log(1)\right) \\ &= \pi_t \log(C^e_{it}) + (1-\pi_t)\log(C^u_{it}) + \theta \pi_t \log(1-h_0) \\ &= \pi_t \log(C^e_{it}) + (1-\pi_t)\log(C^u_{it}) + \theta \frac{\log(1-h_0)}{h_0} h_t \end{aligned}$$

- Households have access to a insurance contract that pays one unit of good in unemployed, 0 if employed.
- They buy A_{it} units of insurance, at unit price τ_t

$$\begin{cases} C^e_{it} + \tau_t A_{it} + K^e_{it+1} \leqslant (z_t + 1 - \delta)K_{it} + w_t h_0 & \text{if she works} \\ C^u_{it} + \tau_t A_{it} + K^u_{it+1} \leqslant (z_t + 1 - \delta)K_{it} + A_{it} & \text{if not} \end{cases}$$

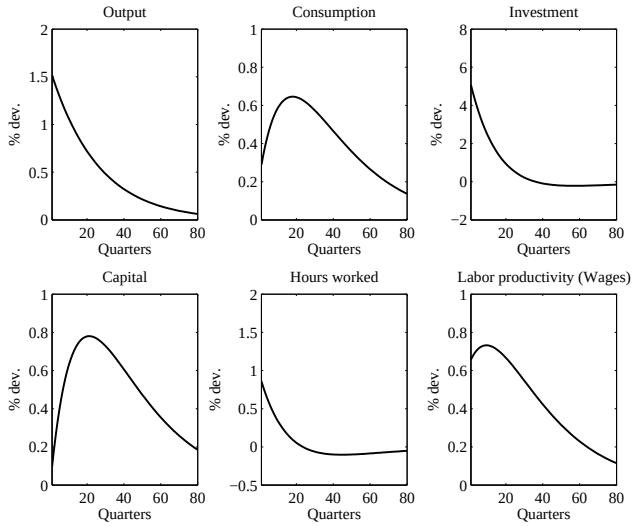
- Households cannot manipulate the probability of working
- Risk neutral insurance companies will make zero profit : Revenues $\tau_t A_t$ = Costs $(1 - \pi_t)A_t$
- Therefore $\tau_t = 1 - \pi_t$
- Households will demand insurance so as to equalize marginal utility of consumption in the two states (employed and unemployed)
- Therefore $C^e_t = C^u_t$ (because utility is additively separable in consumption and leisure)

- We have (assuming they make same accumulation choice) :

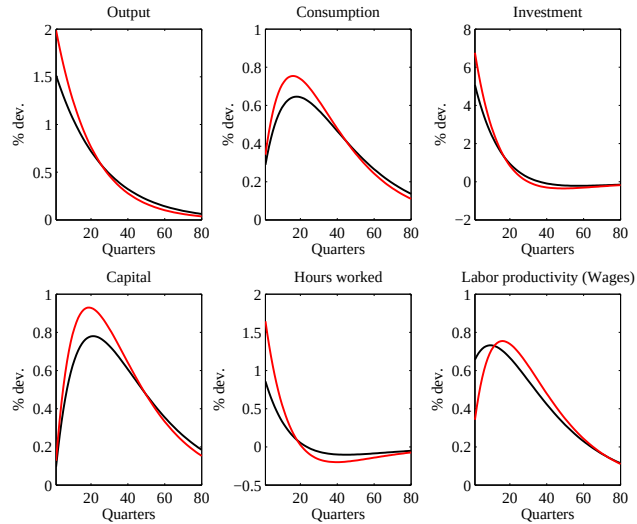
$$\begin{cases} C_t^e = -K_{t+1} + (z_t + 1 - \delta)K_{it} + w_t h_0 - (1 - \pi_t)A_t & \text{if she works} \\ C_t^u = -K_{t+1} + (z_t + 1 - \delta)K_{it} + \pi_t A_{it} & \text{if not} \end{cases}$$

- Equalization of C^u and C^e gives $A = wh_0$.
- As consumption is the same and utility separable, accumulation decisions will be similar
- Assume that agents had initially the same wealth (in period 0) : they will always take similar consumption and saving decisions
- An agent utility in t can therefore be written
  $\pi_t(\log C_t + \theta \log(1 - h_0)) + (1 - \pi_t)(\log C_t + \theta \log 1)$
- Utility collapses to
  $$U(C_t, 1 - h_t) = \log(C_t) - Bh_t$$
  with $B = \theta \frac{\log(1-h_0)}{h_0}$
- $\rightsquigarrow$ A lot (infinite) of labor supply elasticity
- The rest of the model (firms, equilibrium equations) remains unchanged

IRF to A Technological Shock - Indivisible Labor Model



IRF to A Technological Shock - Indivisible Labor Model



Indivisible Labor Model

Variable	$\sigma(\cdot)$	$\sigma(\cdot)/\sigma(y)$	$\rho(\cdot, y)$	$\rho(\cdot, h)$	
Output	1.70	–	–	–	0.84
	1.95	–	–	–	0.68
Consumption	0.80	0.47	0.78	–	0.83
	0.45	0.23	0.78	–	0.83
Investment	6.49	3.83	0.84	–	0.81
	6.66	3.40	0.99	–	0.67
Hours worked	1.69	1.00	0.86	–	0.89
	1.63	0.83	0.98	–	0.67
Labor productivity	0.90	0.53	0.41	0.09	0.69
	0.45	0.23	0.77	0.65	0.83

(model in gray)

- Chari, Kehoe and McGrattan, "Business Cycle Accounting", Econometrica, Vol. 75, No. 3, 2007
- Notation : s_t are events, s^t are histories, $\pi_t(s^t)$ is probability of s^t as of period 0.
- Four wedges, that are four exogenous stochastic variables functions of the underlying random variable s^t
  1. Efficiency wedge $A_t(s^t)$
  2. Labor wedge $1 - \tau_{\ell t}(s^t)$
  3. Investment wedge $1/[1 + \tau_{xt}(s^t)]$
  4. Government consumption wedge $g_t(s^t)$

- Consumers maximize

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) U(c_t(s^t), \ell_t(s^t)) N_t$$

subject to

$$\begin{aligned} c_t(s^t) + [1 + \tau_{xt}(s^t)] x_t(s^t) &= [1 - \tau_{lt}(s^t)] w_t(s^t) \ell_t(s^t) + r_t(s^t) k_t(s^{t-1}) \\ &\quad + T_t(s^t) \\ (1 + \gamma_n) k_{t+1}(s^t) &= (1 - \delta) k_t(s^{t-1}) + x_t(s^t) \end{aligned}$$

- Firms maximize

$$A_t(s^t) F(k_t(s^{t-1}), (1 + \gamma)^t \ell_t(s^t)) - r_t(s^t) k_t(s^{t-1}) - w_t(s^t) \ell_t(s^t).$$

- Equilibrium is summarized by

$$
\begin{aligned}
y_t(s^t) &= c_t(s^t) + x_t(s^t) + g_t(s^t), && (32)\\
y_t(s^t) &= A_t(s^t)F(k_t(s^{t-1}), (1+\gamma)^t \ell_t(s^t)), && (33)\\
\frac{U_{\ell t}(s^t)}{U_{ct}(s^t)} &= [1-\tau_{\ell t}(s^t)]A_t(s^t)(1+\gamma)^t F_{\ell t}, && (34)
\end{aligned}
$$

$$
\begin{aligned}
&U_{ct}(s^t)[1+\tau_{xt}(s^t)] = \\
&\beta \textstyle\sum_{s^{t+1}} \pi_t(s^{t+1}|s^t) U_{ct+1}(s^{t+1}) \left\{ A_{t+1}(s^{t+1}) F_{kt+1}(s^{t+1}) \right. && (35)\\
&\left. +(1-\delta)[1+\tau_{xt+1}(s^{t+1})] \right\}
\end{aligned}
$$

- Note that one wedge (at least) enters in each of the four equations.

- Those wedges are reduced forms for many imperfections :

  Efficiency Wedge : Technological regression (?), mismeasurement, disorganization
  Labor Wedge : Taxes, Unions, wages rigidities, implicit contracts, ...
  Investment Wedge : Credit market imperfections,
  Government Wedge : Government spending shocks, net exports shocks

- Consider (y_t, c_t, ℓ_t, x_t).
- One can always find a sequence of the four wedges such that the model outcome exactly matches observed (y_t, c_t, ℓ_t, x_t)
- The procedure is implemented on the US Great Depression
- As the government wedge plays no role for y, x and ℓ, it is set to zero and we only consider fluctuations in y, x and ℓ.

- The 3 wedges are measured so that one exactly reproduces the movements of y, x and ℓ

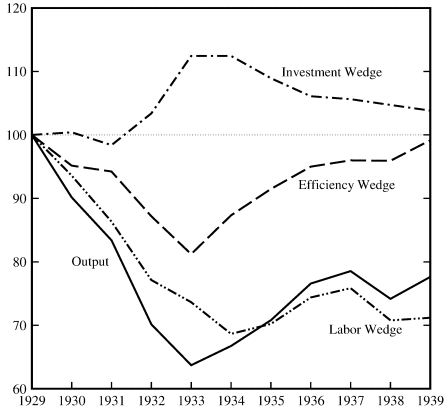
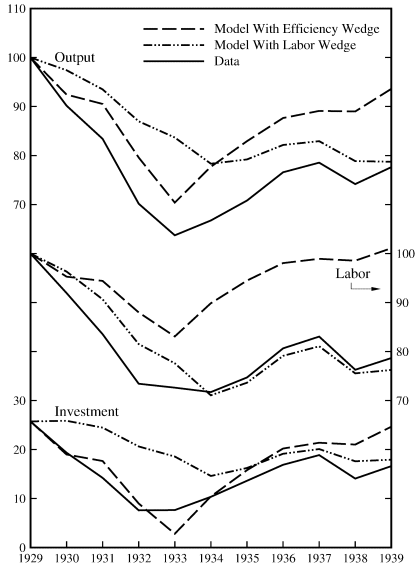


FIGURE 1.—U.S. output and three measured wedges (annually; normalized to equal 100 in 1929).

- The efficiency wedge is in 1933 17 % below the 1929 level, but is back to normal level by 1939.
- The labor wedge remains 30% below (permanently ?)
- The investment wedge goes the wrong way
- CKM look at the fraction of macroeconomic movements that are explained by those 2 wedges, then by only the investment one, then by all but one wedge :
- Warning : Those wedges are NOT orthogonal.

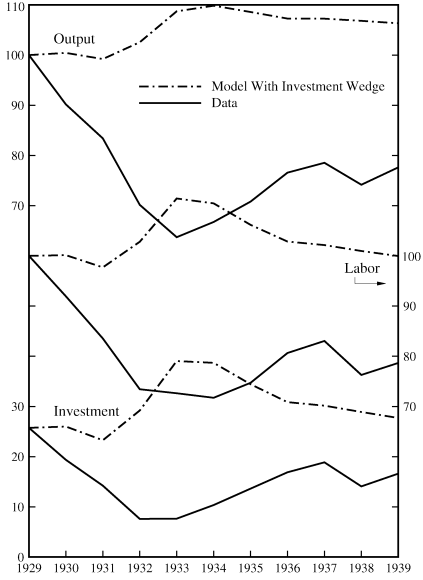
4. Quantitative Evaluation of The Standard RBC Model

Business Cycle Accounting



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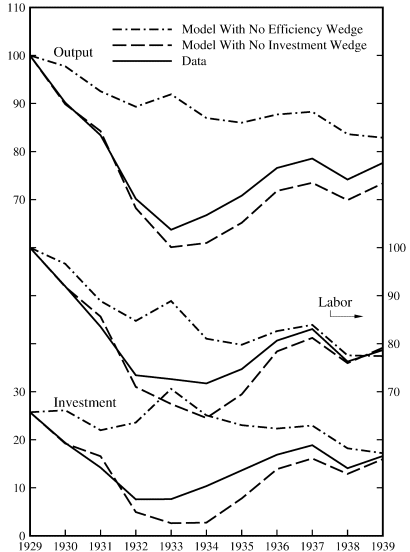


FIGURE 4.—Data and predictions of the models with all wedges but one.

- With the efficiency wedge only : much of the downturn is accounted for, bit the recovery is too important
- The efficiency wedge misses labor market outcome
- With the labor market wedge only, the downturn is partially missed, but the absence of recovery is well accounted for.
- labor market outcomes are well accounted by the labor wedge.
- With the two wedges, one gets almost all of the action (the investment wedge is not so important)

- *Cole and Ohanian, "New Deal Policies and the Persistence of the Great Depression : A General Equilibrium Analysis," Journal of Political Economy, 2004*
- The recovery from the Great Depression was weak.
- Puzzling : the large negative shocks that some economists believe caused the 1929-33 downturn - including monetary shocks, productivity shocks, and banking shocks - become positive after 1933.
- Thesis : Roosevelt's "New Deal" cartelization policies, which limited competition in product markets and increased labor bargaining power, kept the economy depressed after 1933.
- New Deal $\rightsquigarrow$ National Industrial Recovery Act (NIRA) :
  1. suspended antitrust law and permitted collusion in some sectors...
  2. provided that industry raised wages above market clearing levels and accepted collective bargaining with independent labor unions.

- Obviously, the way we assign fluctuations to wedges is model dependent.
- This is in particular true for the efficiency wedge (TFP)
- Let's illustrate that for the Great Depression
- Replace *disembodied* technical progress by *embodied* one
- Assume that technology is now given by

$$Y_t = AH_t^{\alpha}(z_t J_t)^{1-\alpha}$$

where J is the effective capital stock and A is now constant.

- According to the embodiment assumption, capital J accumulates according to

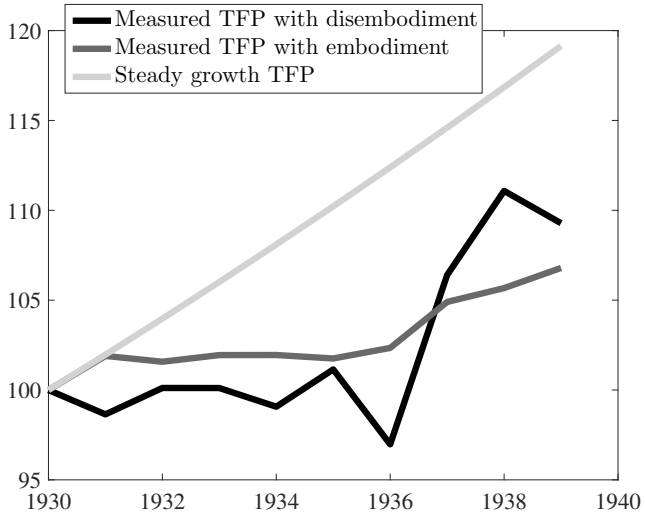
$$J_{t+1} = (1-\delta_J)J_t + X_t I_t$$

where I_t is the National Accounting measure of investment and X a technological factor that grows at rate γ_X.

- If investment is low (for any non technological reason), then *measured* TFP will slowdown

4. Quantitative Evaluation of The Standard RBC Model

Business Cycle Accounting



(...)