2018-2019 – 3-hours lecture @ Cornell

Cycles in Business Cycles

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Note

- Based on my work with Paul Beaudry & Dana Galizia

Objective of this lecture

- Introduce spectral analysis as a useful way to study cycles
- Present some new evidence of (US) cycles
- Present some tools of non-linear dynamics that might help studying cycles

Part I : Business Cycle Definition and Measurement

Part II : Revisiting the Evidence for Cycles

Part III : Generating Cycles through Dynamic Models with Interactions

Part I : Business Cycle Definition and Measurement

Part II : Revisiting the Evidence for Cycles

Part III : Generating Cycles through Dynamic Models with Interactions

- Economists makes a distinction (at least at first pass) between the long run and the short run, between Growth and Business Cycle
- I will look at the U.S.A. as an example.

- Burns and Mitchell *"Measuring Business Cycles"* (1946, National Bureau of Economic Research) :

  "Business cycles are a type of fluctuation found in the aggregate economic activity of nations that organize their work mainly in business enterprises : a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle."

- To identify cycles, B&M assume that they are no shorter than 6 quarters, and found a maximum length of 32 quarters.

FIGURE 1 – Reproduced from Burns and Mitchell *Measuring Business Cycles* (1946)

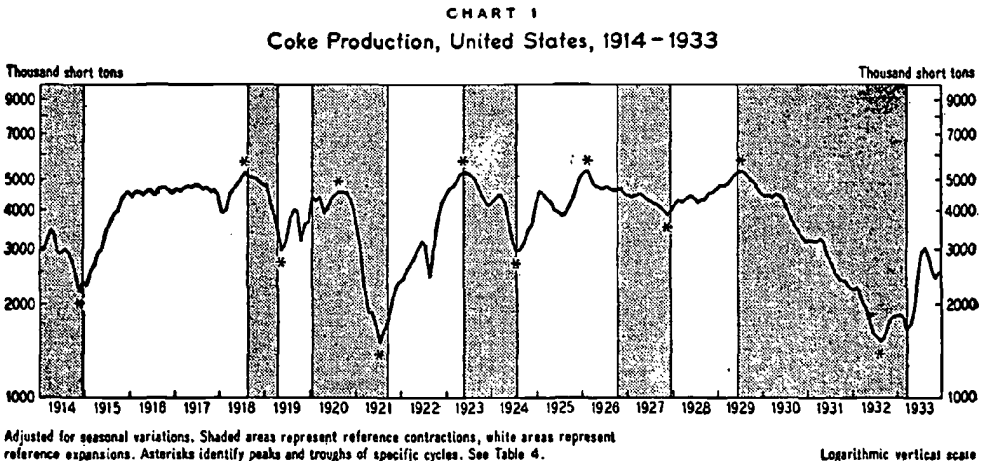


FIGURE 2 – US log Real GDP per capita (trending)

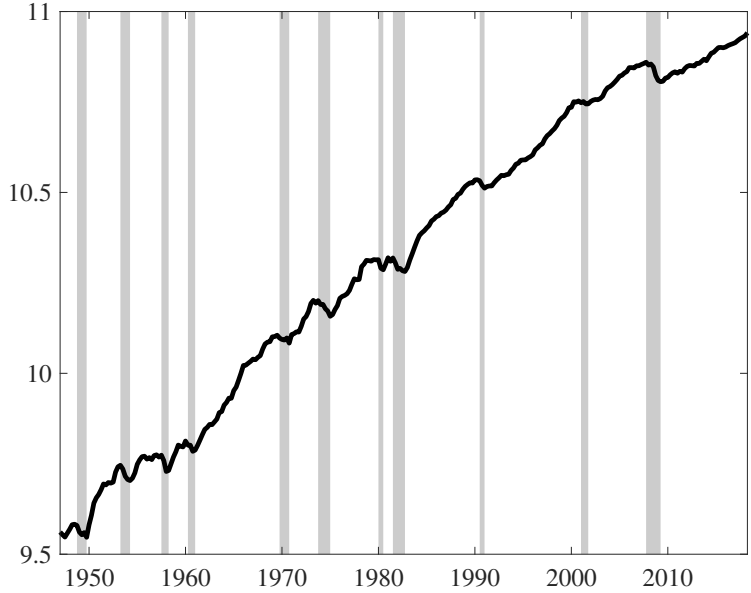


FIGURE 3 – US hours worked per capita (not trending)

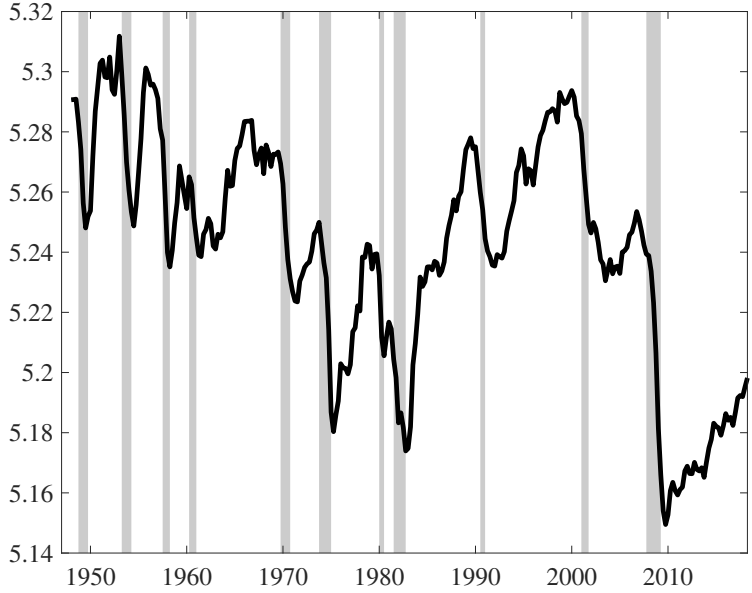


TABLE 1 – Recent U.S. Business Cycles, as identified by the NBER's Business Cycle Dating Committee

BUSINESS CYCLE REFERENCE DATES		DURATION IN MONTHS			
Peak	Trough	Contraction	Expansion	Cycle	
Quarterly dates are in parentheses		*Peak to Trough*	*Previous trough to this peak*	*Trough from Previous Trough*	*Peak from Previous Peak*
April 1960(II)	February 1961 (I)	10	24	34	32
December 1969(IV)	November 1970 (IV)	11	106	117	116
November 1973(IV)	March 1975 (I)	16	36	52	47
January 1980(I)	July 1980 (III)	6	58	64	74
July 1981(III)	November 1982 (IV)	16	12	28	18
July 1990(III)	March 1991(I)	8	92	100	108
March 2001(I)	November 2001 (IV)	8	120	128	128
December 2007 (IV)	June 2009 (II)	18	73	91	81

- How does the NBER establish this chronology ?
- Here is the "Statement of the NBER Business Cycle Dating Committee on the Determination of the Dates of Turning Points in the U.S. Economy".

  "The NBER's Business Cycle Dating Committee maintains a chronology of the U.S. business cycle. The chronology comprises alternating dates of peaks and troughs in economic activity. A recession is a period between a peak and a trough, and an expansion is a period between a trough and a peak. During a recession, a significant decline in economic activity spreads across the economy and can last from a few months to more than a year. Similarly, during an expansion, economic activity rises substantially, spreads across the economy, and usually lasts for several years."

"In both recessions and expansions, brief reversals in economic activity may occur – a recession may include a short period of expansion followed by further decline ; an expansion may include a short period of contraction followed by further growth. The Committee applies its judgment based on the above definitions of recessions and expansions and has no fixed rule to determine whether a contraction is only a short interruption of an expansion, or an expansion is only a short interruption of a contraction . The most recent example of such a judgment that was less than obvious was in 1980-1982, when the Committee determined that the contraction that began in 1981 was not a continuation of the one that began in 1980, but rather a separate full recession."

- Is there a way to translate this into some statistical procedure ?

I.1. A First Look at Some Methods To Extract the Cycle

- Any time series $y_t = \log Y_t$ can be decomposed such that

$$y_t = y_t^T + y_t^C$$

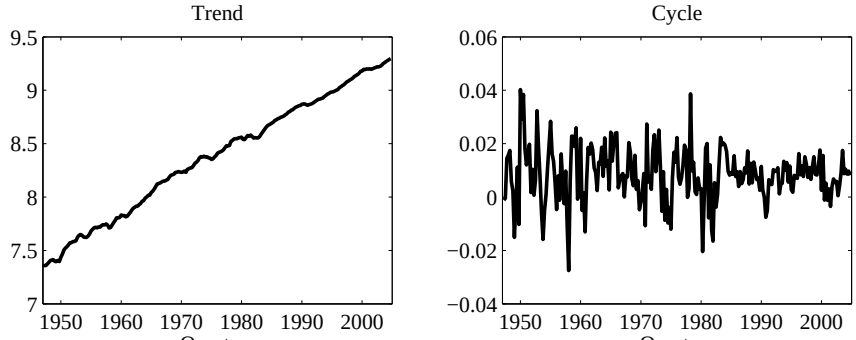
- Problem : How to define/identify each component ?
- Several ways of approaching the problem
- Actually : Infinite number of decomposition of a non-stationary process into a cycle and a trend
- Let us see some "intuitive" definition of those decompositions

I.1. A First Look at Some Methods To Extract the Cycle

Growth Cycle

- Take the growth rate of the series
- Expansion : Positive rate of growth
- Note : the cycle is very volatile (almost iid) – a lot of medium run fluctuations are eliminated

FIGURE 4 – US Growth Cycles



- Deviation from linear trend
- The trend is obtained from linear regression

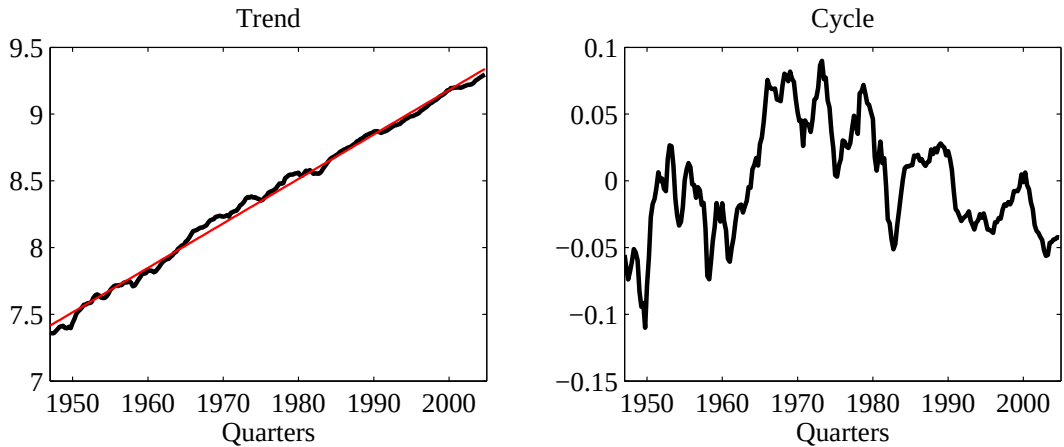
$$y_t = \alpha + \beta t + u_t$$

- Cycle : $y_t^C = y_t - (\widehat{\alpha} + \widehat{\beta} t)$
- Expansion : Output above the trend

I.1. A First Look at Some Methods To Extract the Cycle

Trend Cycle

FIGURE 5 – US Trend Cycles



I.1. A First Look at Some Methods To Extract the Cycle

Cycle = Output Gap

- Define the Output gap as

$$\text{Actual output} - \text{Potential Output}$$

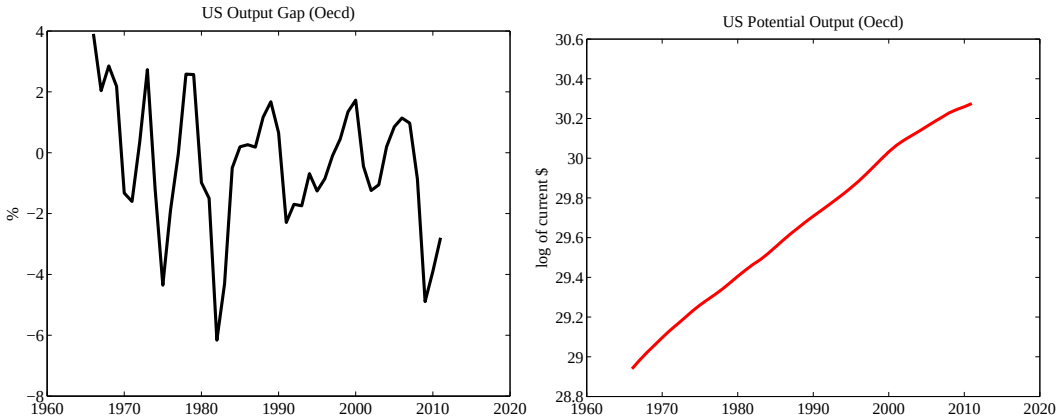
- Expansion : Actual output $\geqslant$ Potential output
- Actual output : easy to observe
- Note : How to identify potential output ? (full utilization ?, efficient ?)
- Example :
  1. estimate $y_t = \alpha \times u_t + \text{other controls} + \varepsilon_t$,
  2. define potential output as $y_t^P = \widehat{\alpha} \times 0\% + \text{other controls} + \widehat{\varepsilon}_t$. (One might choose $u = u^n$ where u^n is the *natural* rate of unemployment (the Oecd chooses the NAIRU (Non Accelerating Inflation Rate of Unemployment))
- Cycle is then $y_t - y_t^P$

I.1. A First Look at Some Methods To Extract the Cycle

Cycle = Output Gap

- This is an over simplified description of the method used by Oecd.

FIGURE 6 – US Output Gap and Potential Output



I.1. A First Look at Some Methods To Extract the Cycle

- One could describe many other methods to extract the Business Cycle.
- Ideally, we want to get rid of very short run and long run movements of economic activity.
- The best way to understand this is to decompose economic time series into the frequency domain and filter them.
- For this, we need to understand how a time series can be represented in the frequency domain.
- Here I am giving the main intuitions, and will not be very rigourous. (see some stats or metrics courses)

I.2. Decomposing a time series into frequency domain

Background

- A (second order) stationary time series x_t with $E[x_t] = 0$ has three fundamental representations :
- 1. Autocovariance function

$$\lambda_\tau = E[x_t x_{t-\tau}]$$

- 2. MA representation (Wold theorem) :

$$x_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \qquad \text{and} \qquad \lambda_\tau = \left(\sum_{j=0}^{\infty} \theta_j \theta_{j+\tau} \right) \sigma_\varepsilon^2$$

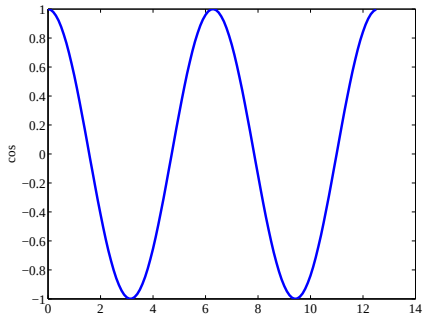
- 3. Spectral representation :

$$s(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{+\infty} \lambda_\tau (\cos(\omega\tau) - i \sin(\omega\tau))$$

- Note : works only for stationary series

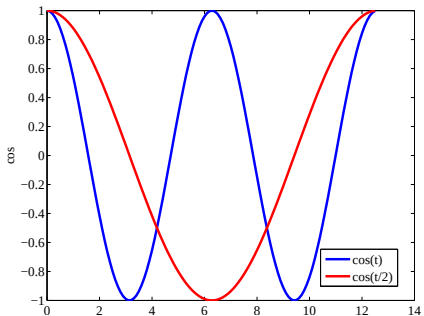
- Idea : A series can be seen as the sum of periodic functions.
- A typical periodic function is $\cos(\omega t)$, with period (the time it takes to reproduce itself) $2\pi/\omega$.
  - × Knowing that period of $\cos(t)$ is 2π, for a given t_1, what is the t_2 such that $\cos(\omega t_2) = \cos(\omega t_1)$?
  - × The solution is $t_2 - t_1 = 2\pi/\omega$.
- $\frac{\omega}{2\pi}$ is the *frequency* of oscillation (number of cycles per unit of time)

FIGURE 7 – Cosine wave with ω=1



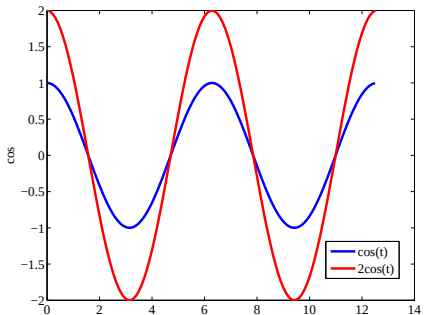
- With $\omega = 1$, the period is $2\pi = 6.28$ and frequency is $\frac{1}{2\pi} = 0.16$.

FIGURE 8 – Cosine waves with ω=1 or 1/2



- With $\omega = 1/2$, the period is $4\pi = 12.56$ and frequency is $\frac{1}{4\pi} = 0.08$.

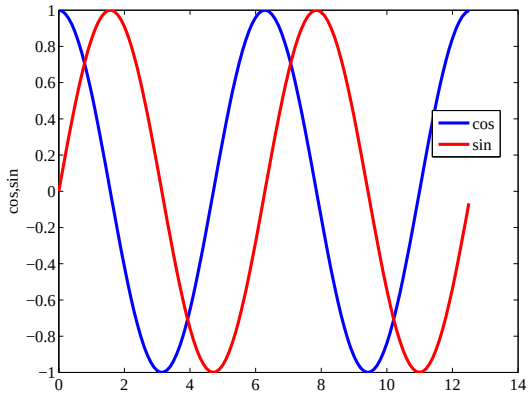
FIGURE 9 – Cosine waves with ω=1 and different amplitudes



- Here are plotted $A\cos(t)$ with $A = 1$ or $A = 2$.

- $\sin(\omega t)$ behaves the same way, with same amplitude and period, but with a phase shift

FIGURE 10 – Cosine and Sine waves with ω=1



- The idea of spectral decomposition is that with sin and cos, we can span the whole space of covariance stationary time series : the typical periodic function is

$$a\cos(\omega t) + b\sin(\omega t) \tag{1}$$

  whose period is $2\pi/\omega$ but whose phase and amplitude depend on (a, b)
- Here we want to treat a and b as mean zero random variables.
- There is always a sum of type (1) periodic functions that reproduces a given time series
- The spectral density or *spectrum* of a series indicates the weight of each frequency (from low to high) in the total variance of the series.

- Assume that we observe y_t over T (even) periods, and that it is centered.
- Our goal is to decompose y_t into $T/2$ periodic functions of frequencies $\omega_1, \omega_2, ..., \omega_{T/2}$, with

$$\omega_j = \frac{2\pi j}{T}, \quad j = 1, ..., T/2$$

- Then, we want to write y_t as

$$\begin{aligned} y_t &= a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t) \\ &+ a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t) \\ &+ \cdots \\ &+ a_{T/2} \cos(\omega_{T/2} t) + b_{T/2} \sin(\omega_{T/2} t) \end{aligned} \tag{2}$$

for t=1,...,T.

- Find the T parameters (a_i, b_i) in (2) by OLS.

$$X = \begin{bmatrix} \cos(\omega_1) & sin(\omega_1) & \cdots\cdots & \cos(\omega_{T/2}) & sin(\omega_{T/2}) \\ \cos(2\omega_1) & sin(2\omega_1) & \cdots\cdots & \cos(2\omega_{T/2}) & sin(2\omega_{T/2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos(T\omega_1) & sin(T\omega_1) & \cdots\cdots & \cos(T\omega_{T/2}) & sin(T\omega_{T/2}) \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{T-1} \\ y_T \end{bmatrix} \quad \beta = \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ \vdots \\ a_{T/2} \\ b_{T/2} \end{bmatrix}$$

- If we assume $Y = X\beta + u$, we can compute the a and b.

- The a and b coefficients are computed as

$$\beta = (X'X)^{-1}X'Y$$

- Given that we have T explanatory variables for T observations, the R^2 is one and $u = 0$. Here we are just solving a *representation* problem, not an estimation one.
- Note that the last column of X is a column of 0. It is replaced by a column of 1 to deal with non-centered series.

- The coefficient are given by

$$a_j = \frac{2}{T}\sum_{t=1}^{T}\cos(\omega_j t)y_t \quad (3)$$

$$b_j = \frac{2}{T}\sum_{t=1}^{T}\sin(\omega_j t)y_t \quad (4)$$

for $j \leq T/2 - 1$, and

$$a_{T/2} = \frac{1}{T}\sum_{t=1}^{T}\cos(\omega_{T/2} t)y_t \quad (5)$$

$$b_{T/2} = \frac{1}{T}\sum_{t=1}^{T} y_t \quad (6)$$

- Let T goes to infinity.
- The spectral representation is :

$$y_t = \int_0^{\pi} (a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t))d\omega \tag{7}$$

- Any covariance stationary times series process can be represented in the form of (7).
- The function that gives the variance of $a(\omega)$ and $b(\omega)$, as a function of ω is the spectrum.
- Is y deterministic ? No, "as if" $a(\omega)$ and $b(\omega)$ were stochastic

I.2. Decomposing a time series into frequency domain

Extracting the business cycle (BC) component using the frequency domain representation

- A representation like (7) allow us to make precise the notion of extracting the business cycle component of y_t.
- Assume y_t is observed on a quarterly basis, and that the BC is defined as fluctuations of periods between 6 and 32 quarters (1.5 to 8 years), i.e. for $\omega \in [\underline{\omega} \;\; \overline{\omega}] = [\frac{2\pi}{32}, \frac{2\pi}{6}]$.
- Then the BC component of y, denoted y^C, is

$$y_t^C = \int_{\underline{\omega}}^{\overline{\omega}} (a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t))d\omega \tag{8}$$

- It can be shown that this spectral representation has a time-series equivalent, which is an infinite two-sided moving average of y_t :

$$y_t^C = B_0 y_t + B_1(y_{t-1} + y_{t+1}) + B_2(y_{t-2} + y_{t+2}) + \cdots \tag{9}$$

with $B_0 = \frac{\overline{\omega}-\underline{\omega}}{\pi}$ and $B_j = \frac{\sin(\overline{\omega}j)-\sin(\underline{\omega}j)}{\pi j}$

$$y_t^C = \int_{\underline{\omega}}^{\overline{\omega}} (a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t))d\omega \tag{8}$$

$$y_t^C = B_0 y_t + B_1(y_{t-1} + y_{t+1}) + B_2(y_{t-2} + y_{t+2}) + \cdots \tag{9}$$

- This is what we call a Band-Pass filter
- Why things are not as simple as they look ?
- We have to make an approximation of (9) because it requires an infinity of observations.
- Therefore, the MA is truncated according to some distance criterium

- We consider here *univariate stationary processes*.

Definition 1

The *autocovariance* of a series Y_t is defined as $\lambda_\tau = \text{cov}(Y_t, Y_{t-\tau}) = E(Y_t Y_{t-\tau})$ with the assumption $E(Y_t) = 0$;.

Definition 2

For a sequence $a_0, a_1, ..., a_j, ...$, the **generating function** of this sequence is $a(z) = \sum_j a_j z^j$.

- Note : z needs not to have any interpretation
- The generating function (or z-transform) of a process Y_t is $Y(z) = \sum_t Y_t z^t$.

I.2. Decomposing a time series into frequency domain

Spectrum

- Correspondence between spectrum and auto-covariogram :
- Assume $z = e^{-i\omega} = \cos\omega - i\sin\omega$ and define

$$s(\omega) = \frac{1}{2\pi}\lambda(z) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{+\infty}\lambda_\tau e^{-i\omega\tau}$$

- Then one can show that

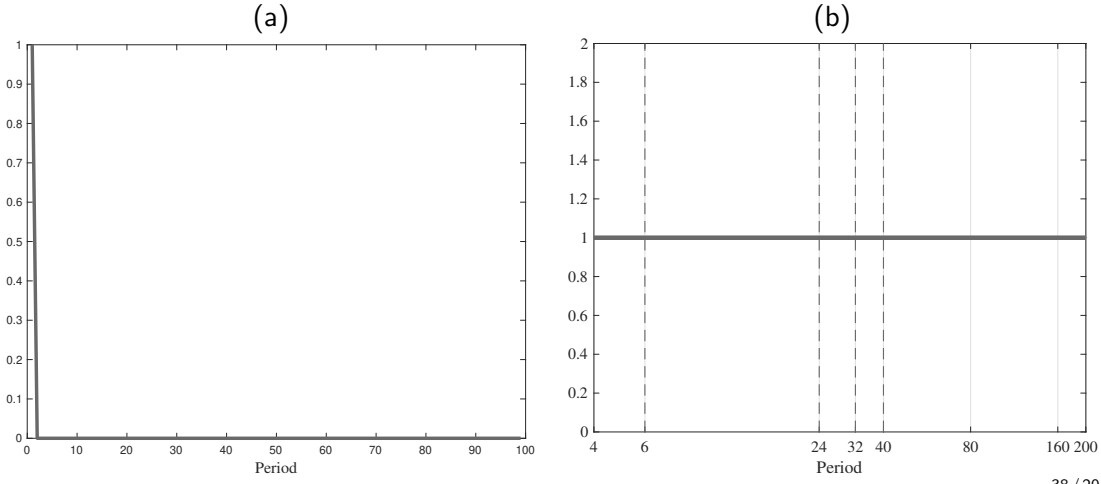
$$\lambda_\tau = \int_{-\pi}^{\pi} e^{i\tau\omega}s(\omega)d\omega \quad \text{and} \quad \lambda_0 = \int_{-\pi}^{\pi} s(\omega)d\omega$$

- The sequence of $\{\lambda_\tau\}$ and $\{s(\omega)\}$ bring the same information.
- The function $\frac{s(\omega)}{\lambda_0}$ has the property of a probability function over $-\pi \leq \omega \leq \pi$: $s(\omega) \geq 0$ and $\int_{-\pi}^{\pi}\frac{s(\omega)}{\lambda_0}d\omega = 1$.
- $s(\omega)$ (rescaled) is the spectral density.

I.2. Decomposing a time series into frequency domain

$x_t = \varepsilon_t$

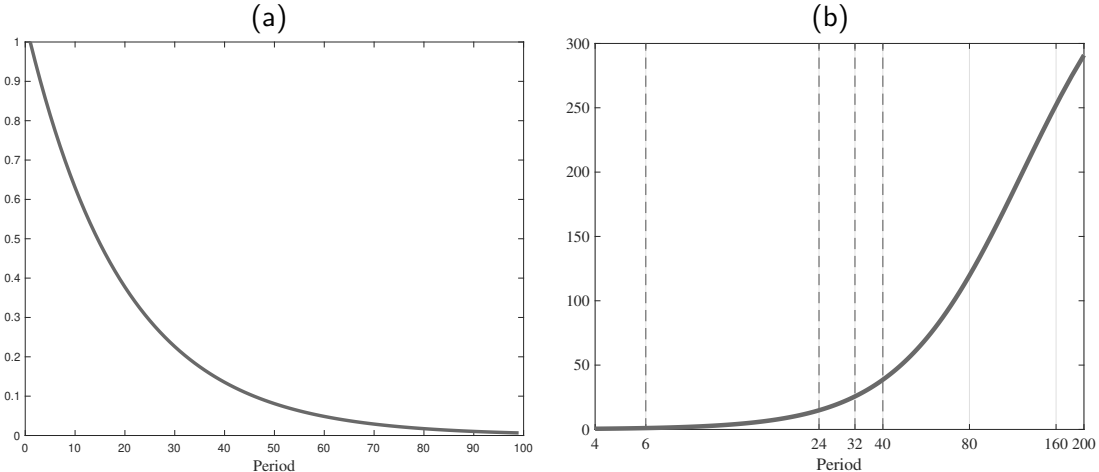
FIGURE 11 – (a) IRF and (b) Spectrum



I.2. Decomposing a time series into frequency domain

$x_t = .95x_{t-1} + \varepsilon_t$

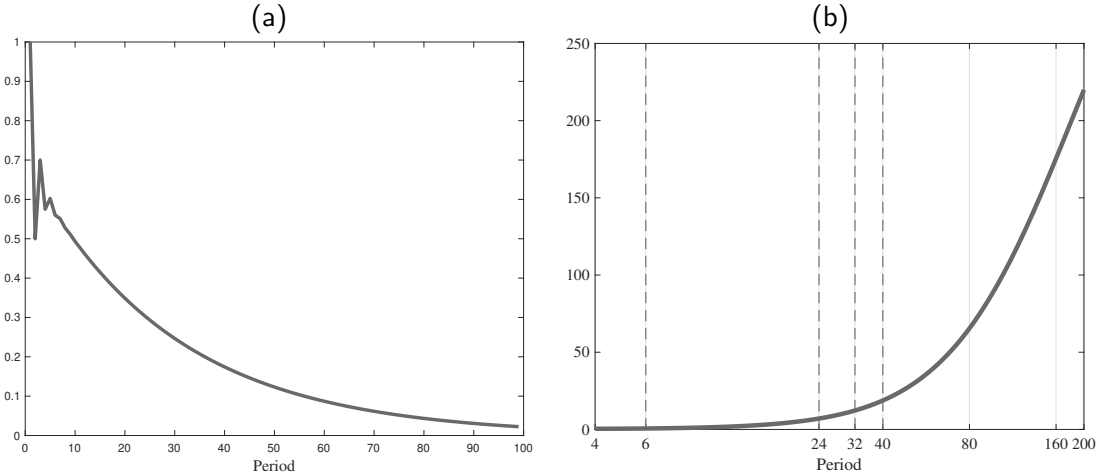
FIGURE 12 – (a) IRF and (b) Spectrum



I.2. Decomposing a time series into frequency domain

$x_t = .5x_{t-1} + .45x_{t-2} + \varepsilon_t$

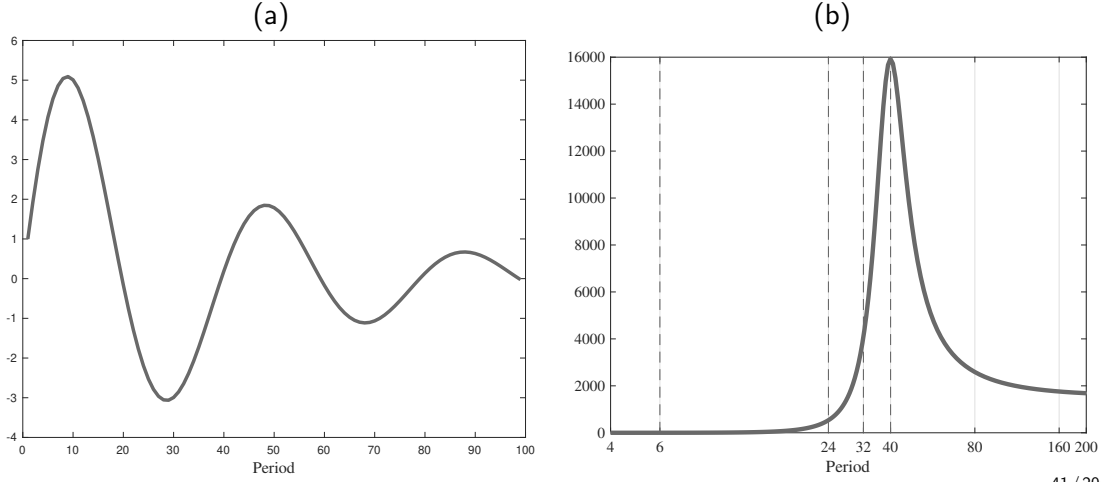
FIGURE 13 – (a) IRF and (b) Spectrum



I.2. Decomposing a time series into frequency domain

$x_t = 1.92x_{t-1} + .95x_{t-2} + \varepsilon_t$

FIGURE 14 – (a) IRF and (b) Spectrum



Definition 3

Let $\widehat{Y}_t = \sum_{j=0}^{m} c_j Y_{t-j} = C(L)Y_t$. $\widehat{Y}_t$ is a **filtered** version of Y_t.

- One can show that the spectral density of $\widehat{Y}_t$ is

$$s_{\widehat{y}}(\omega) = C(e^{-i\omega})C(e^{i\omega})s_y(\omega) = |C(e^{i\omega})|^2 s_y(\omega)$$

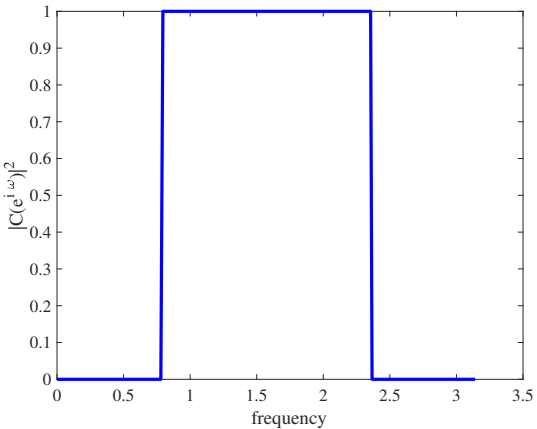
.

- $|C(e^{i\omega})|^2$ is the transfer function of the filter

Exact Band Pass Filter

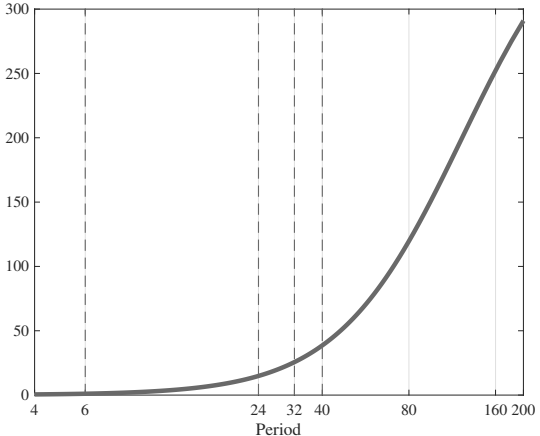
- Corresponds to a two-sided infinite filter.
- $|C(e^{i\omega})|^2 = 1$ if $\omega \in [\underline{\omega} \;\; \overline{\omega}]$ and 0 if not.

FIGURE 15 – Exact Band Pass Filter Transfer Function



- Example : Band Pass Filter applied on the AR(1) process $x_t = .95x_{t-1} + \varepsilon_t$

FIGURE 16 – Spectrum of the AR(1)

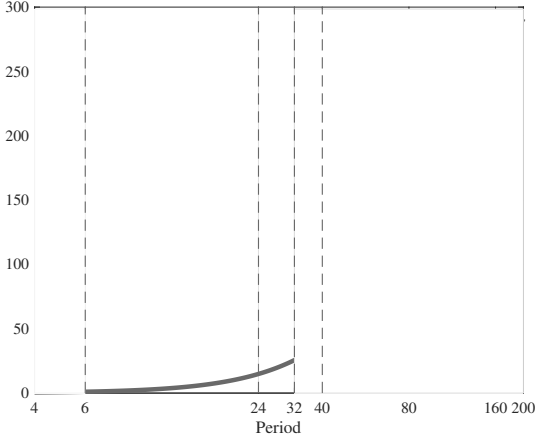


I.2. Decomposing a time series into frequency domain

Exact Band Pass Filter

- Example : Band Pass Filter applied on the AR(1) process $x_t = .95x_{t-1} + \varepsilon_t$

FIGURE 17 – Spectrum of the AR(1), Band passed

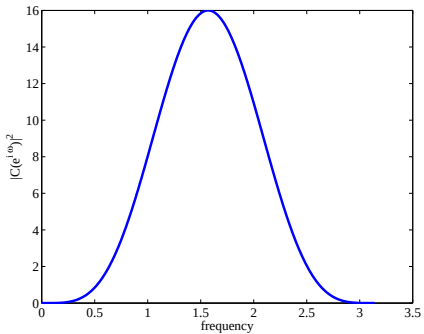


Approximate Band Pass Filter

- $c_0 = 1, c_2 = -2, c_4 = -1$ and $c_j = 0$ for other j.
- $\widehat{Y}_t = C(L)Y_t = Y_t - 2(Y_{t-2} + Y_{t+2}) - (Y_{t-4} + Y_{t+4})$
- Then

$$|C(e^{i\omega})|^2 = 4(1 - \cos 2\omega)^2$$

FIGURE 18 – Approximate Band Pass Filter Transfer Function



- Here is an example of the use of Band Pass filters from Roberto Pancrazi, "Spectral Covariance Instability Test : An Application to the Great Moderation", TSE 2010.
- High Frequency : periodicity between 2 and 32 quarters
- Medium Frequency : periodicity between 32 and 80 quarters
- High to Medium Frequency : periodicity between 2 and 80 quarters

FIGURE 19 – US GDP

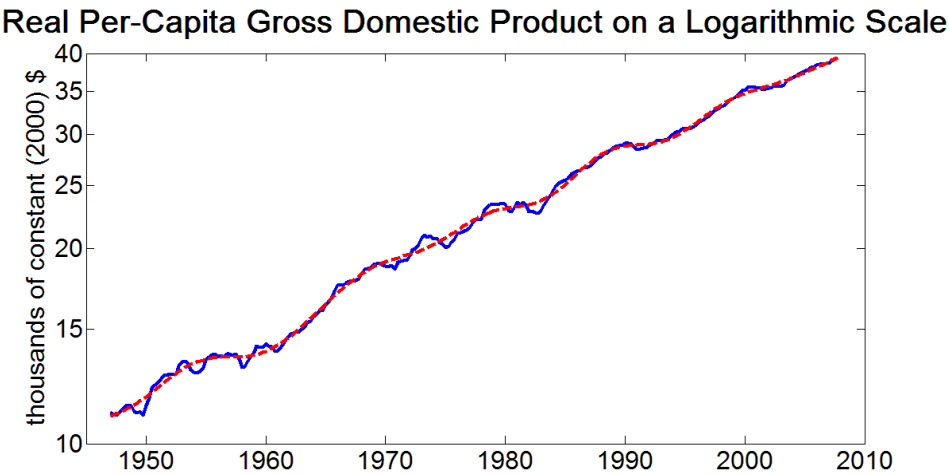
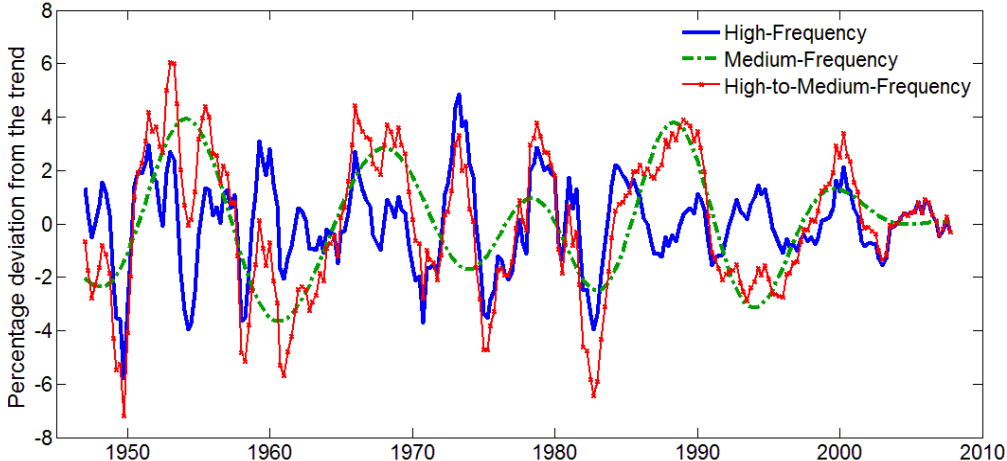


FIGURE 20 – Various Cycles

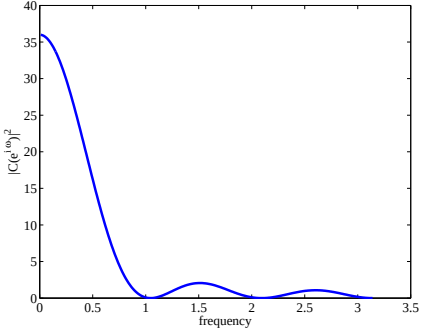


Low Pass Filter

- $\widetilde{Y}_t = \sum_{j=0}^{m} Y_{t-j}$. Then

$$|C(e^{i\omega})|^2 = \frac{1-\cos(m+1)\omega}{1-\cos\omega}$$

FIGURE 21 – A Low Pass Filter Transfer Function

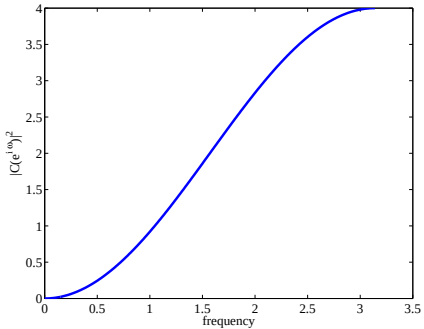


First Difference

- First difference $\widetilde{Y}_t = (1 - L)Y_t$. Then

$$|C(e^{i\omega})|^2 = 2 - 2\cos\omega$$

FIGURE 22 – First Difference Transfer Function



A High Pass Filter : The Hoddrick-Prescott Transfer Function

- Very popular in the macro literature
- In the time domain, the idea is to remove a trend which is smooth, but not linear
- The trend y_t^T is the Argmin of :

$$\sum_{t=1}^{T}(y_t - y_t^T)^2 + \lambda \sum_{t=2}^{T}((y_{t+1}^T - y_t^T) - (y_t^T - y_{t-1}^T))^2$$

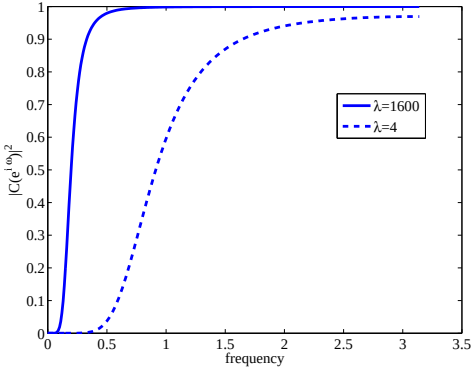
- if $\lambda = +\infty$, it is linear detrending.
- The solution of this program solves
  $y_t/\lambda = y_{t+2}^T - 4y_{t+1}^T + (6+\lambda)y_t^T - 4y_{t-1}^T - y_{t-2}^T$
- The solution is a symmetric MA of order $+\infty$:

$$y_t^T = \sum_{j=-\infty}^{\infty} a_{|j|} y_{t+j}$$

- Then $y_t^C = y_t - y_t^T$ is a time invariant linear symmetric filter.

- With $\lambda = 1600$ on quarterly data, it removes cycles of period greater than 10 years.
- The transfer function is $|C(e^{i\omega})|^2 = \frac{16\lambda^2(1-\cos\omega)^4}{(1+4\lambda(1-\cos\omega)^2)^2}$

Figure 23 – Hoddrick-Prescott Filter



I.2. Decomposing a time series into frequency domain

The HP filter at work

FIGURE 24 – US HP Trend

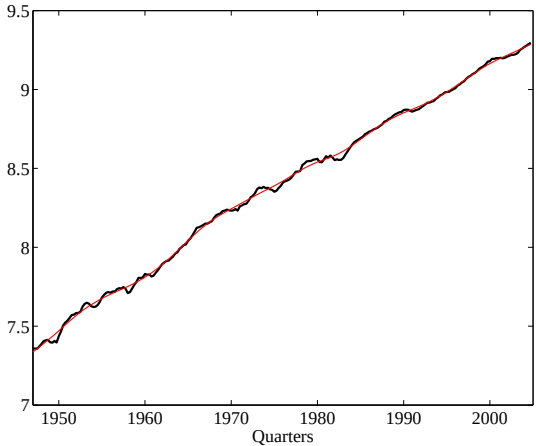
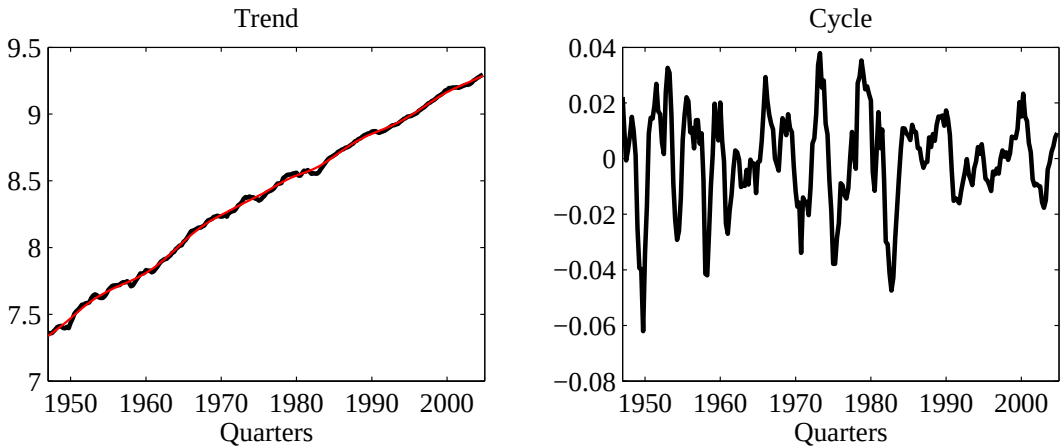
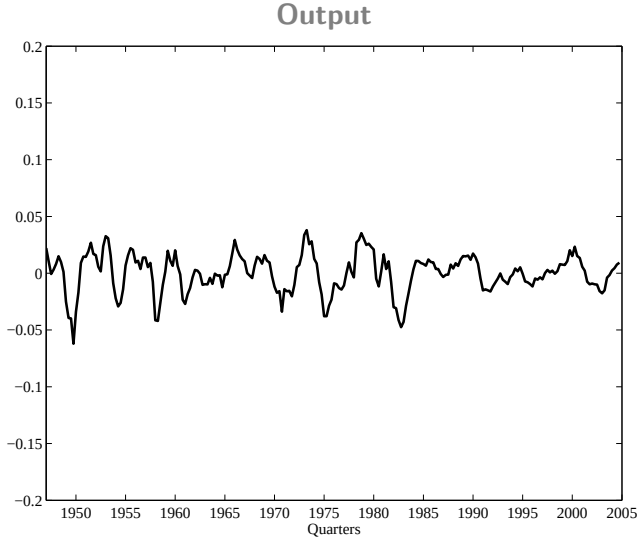


FIGURE 25 – US HP Cycle



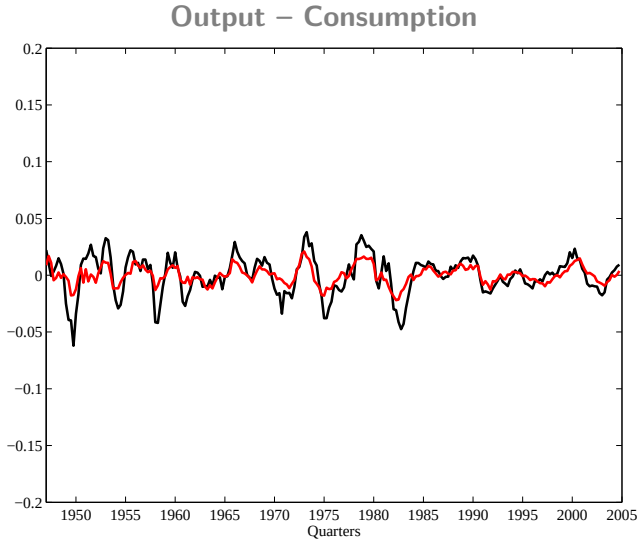
I.2. Decomposing a time series into frequency domain

The HP filter at work



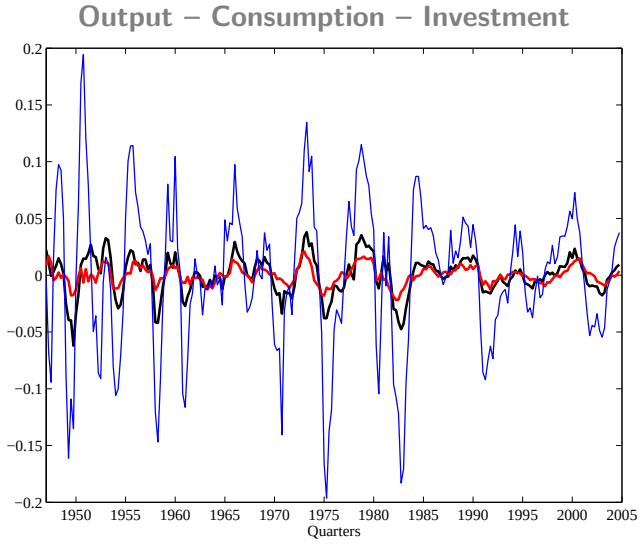
I.2. Decomposing a time series into frequency domain

The HP filter at work



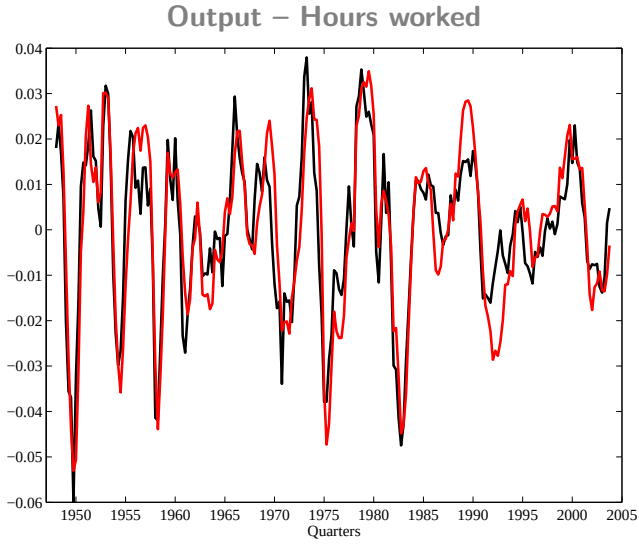
I.2. Decomposing a time series into frequency domain

The HP filter at work



I.2. Decomposing a time series into frequency domain

The HP filter at work



Part I : Business Cycle Definition and Measurement

Part II : Revisiting the Evidence for Cycles

Part III : Generating Cycles through Dynamic Models with Interactions

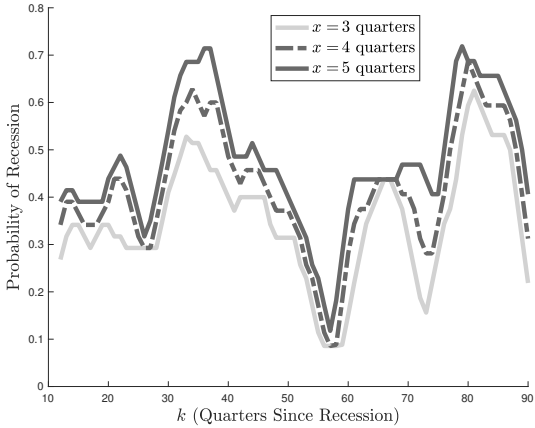
- This part will be empirical with the goal of re-examining some basic questions regarding how we view/frame our discuss about macroeconomic fluctuations (see Part I).
- Questions addressed
  - × What are the main features that business cycle theory should be explaining ?
  - × Could we have been missing one really important element : *cyclicality* ?
  - × How does this relate to the financial cycle ?

- This requires return to the question : what do we mean by a cycle ?
- This has been the focus of Part I.
- At least two different notions
  - × A cycle could mean a tendency for a phenomena, such as a boom and bust (whether real or financial), to recur with a certain degree of regularity
  - × Could alternatively mean : Co-movement features of many variables at a given frequency range
- The later is generally implied for the business cycle (as we have seen in Part I)...
- ...while the former seems to be the current focus for the financial cycle debate (e.g., BIS reports).

- Address three questions
  1. Should we consider business cycle fluctuations as associated with the 6-32 quarters periodicity band or longer ?
  2. Should we consider such fluctuations as mainly trend reverting or cyclical ?
  3. What is the relation with the financial cycle ?
- Focus mainly on US data, but also present some for other rich countries.

II.1. Stationary Variables

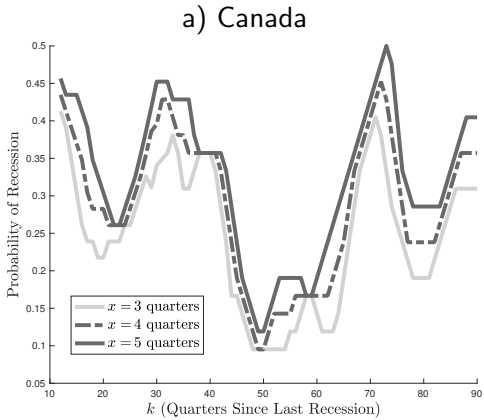
Conditional Probability of Being in a Recession (US)

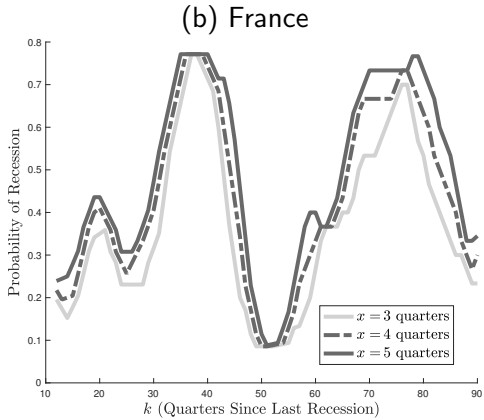


Notes : This shows the fraction of time the economy was in a recession within an x-quarter window around time $t + k$, conditional on being in a recession at time t, where x is allowed to vary between 3 and 5 quarters.

II.1. Stationary Variables

Conditional Probability of Being in a Recession

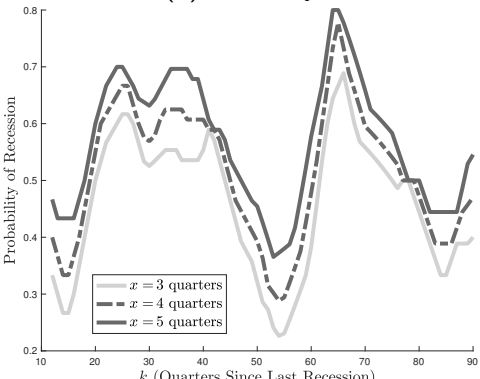




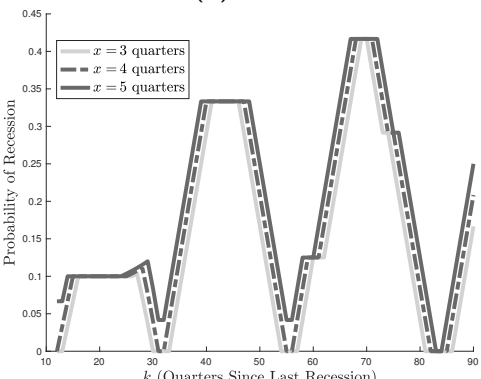
II.1. Stationary Variables

Conditional Probability of Being in a Recession

(a) Germany

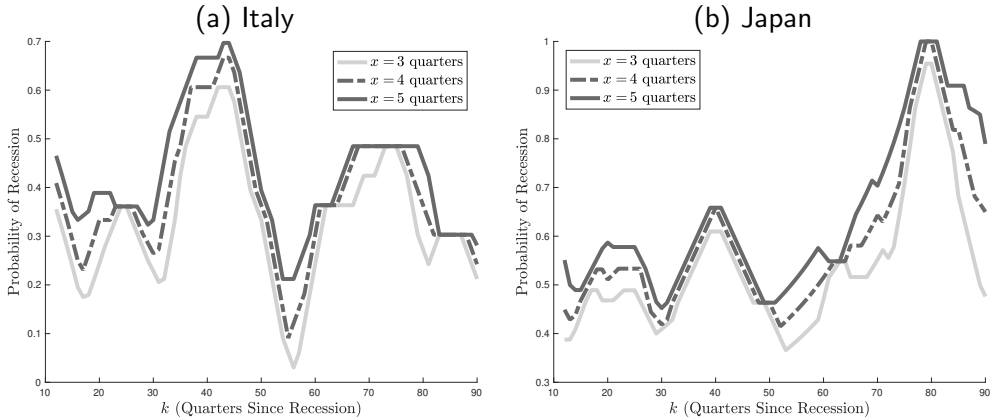


(b) U.K



II.1. Stationary Variables

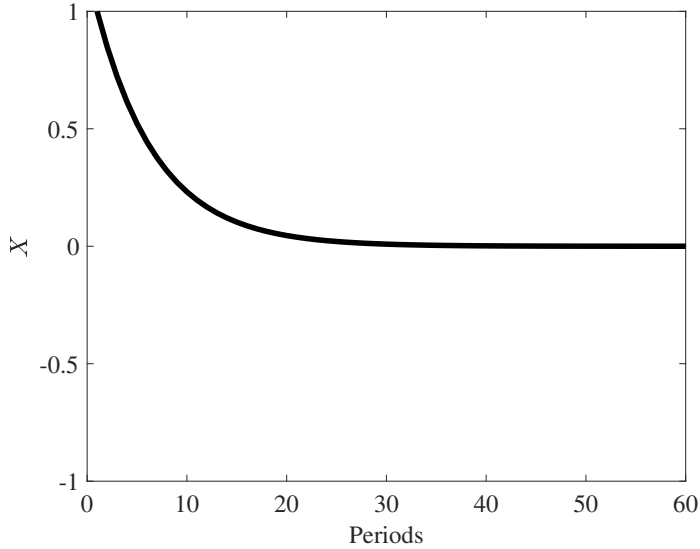
Conditional Probability of Being in a Recession



- What is meant by cyclicality ?
  - × If activity is high today,
  - × at say $N/2$ period in the future, economic activity is expected to be low (below trend),
  - × and then at N expected to be high again and so on.
- This translates in cyclicality in the auto-covariance
- Note : **Nothing deterministic** about this definition, its only about conditional expectations.
- Different from the more standard "auto-regressive" (AR(1)) view.
  - × If activity is high today,
  - × we expect it to return to mean.
- The two views differ whether we should worry about big booms.

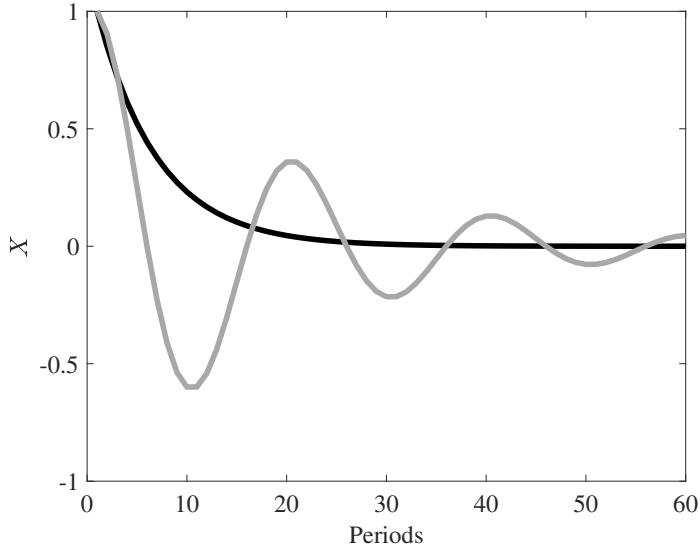
II.1. Stationary Variables

Absence of Cyclicality



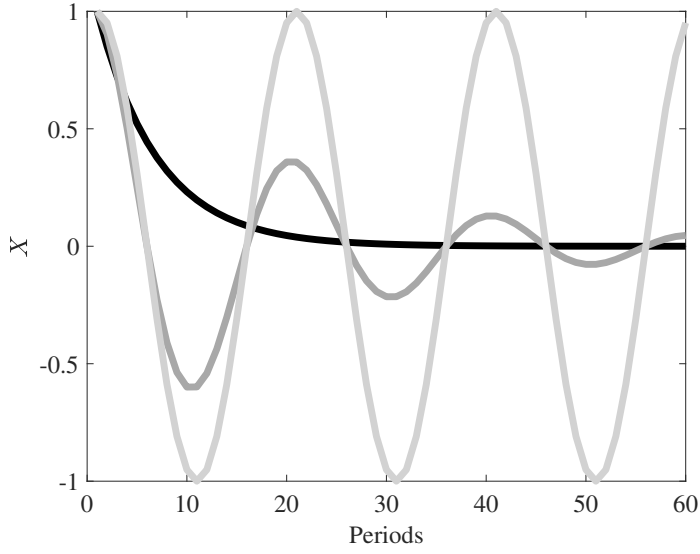
II.1. Stationary Variables

Cyclicality



II.1. Stationary Variables

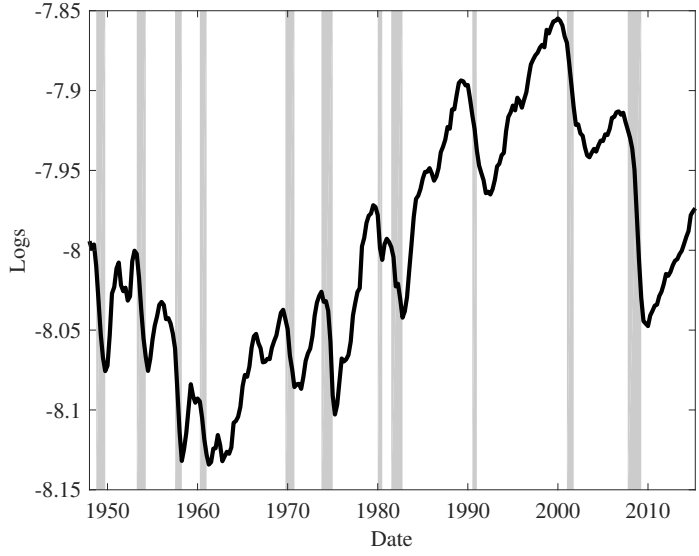
"Strong" Cyclicality



- Issue seems easy to settle : just look at auto-covariance function of the data
- Difficulty : Many macro variables are trending, so this requires a trend-cycle decomposition
- But such decompositions can create spurious cycles
- Solution : Look at hours worked per capita (non trending)

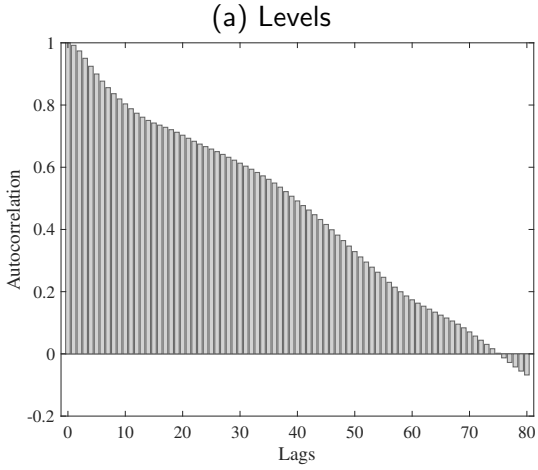
II.1. Stationary Variables

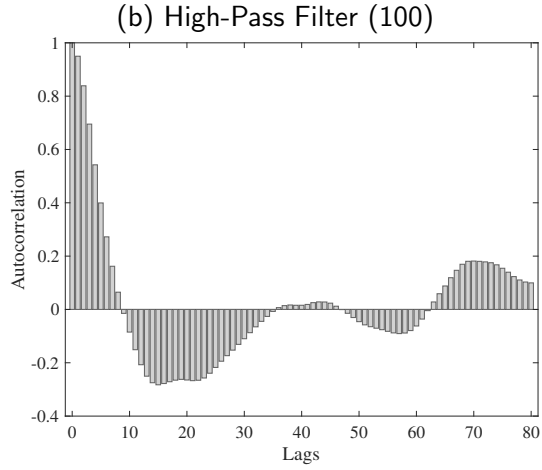
Non-Farm Business (NFB) Hours Per Capita



II.1. Stationary Variables

Autocorrelation of Hours





- A different (better) way to look at cyclicality is to look at spectral density
- There is a one to one mapping between autocovariance and spectral density
- but spectral density "weights" autocorrelations with their contribution to total variance
- $\rightsquigarrow$ autotocorrelations at longer horizons are "boosted" and separated from shorter horizons.

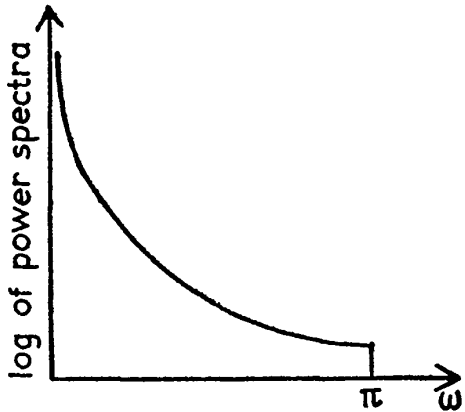


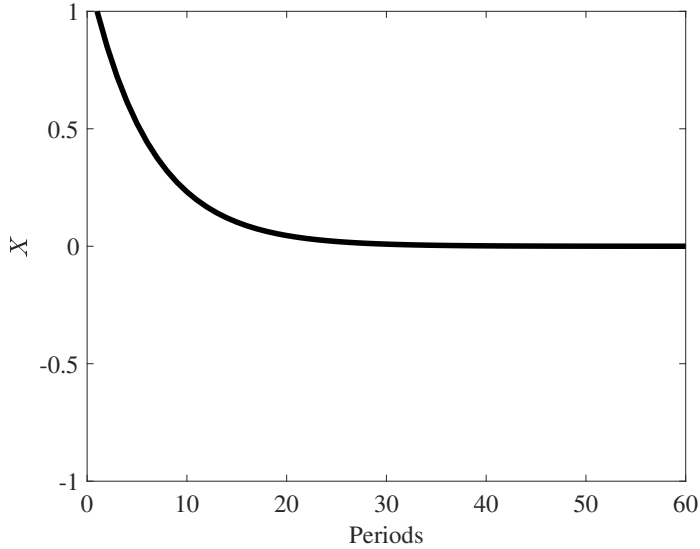
FIGURE 1.—Typical spectral shape.

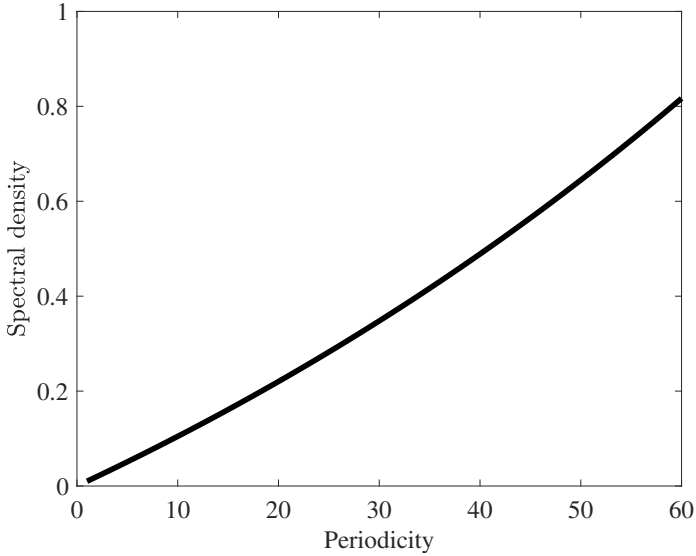
Conventional Wisdom-GRANGER [1969]

FIGURE 1.—Typical spectral shape.

II.1. Stationary Variables

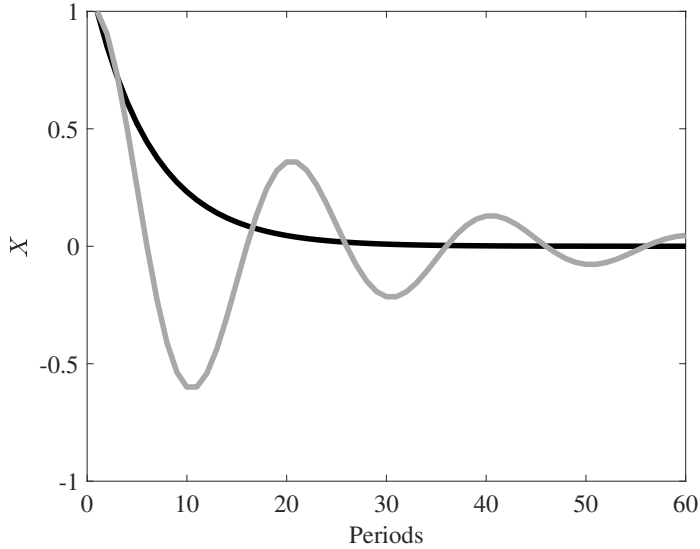
Absence of Cyclicality

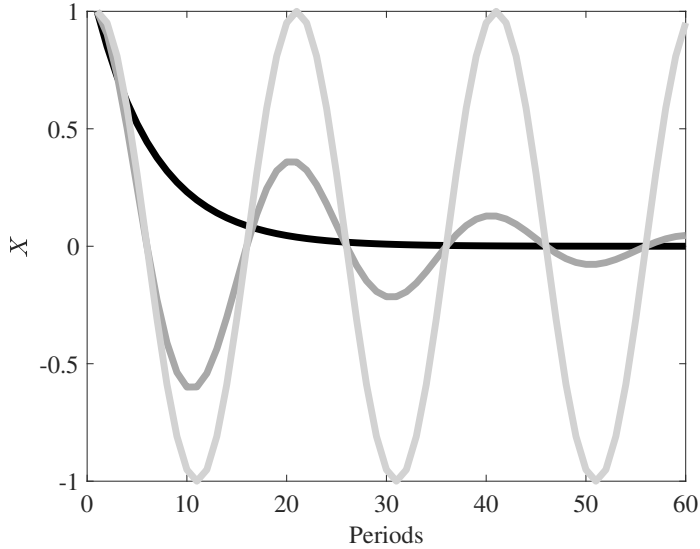



Spectral density
Periodicity
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60

II.1. Stationary Variables

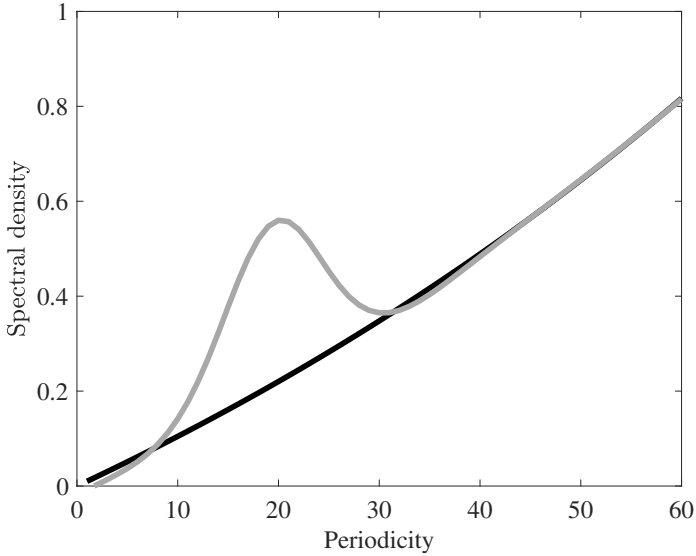
Cyclicality




1
0.5
0
-0.5
-1
X
0
10
20
30
40
50
60
Periods

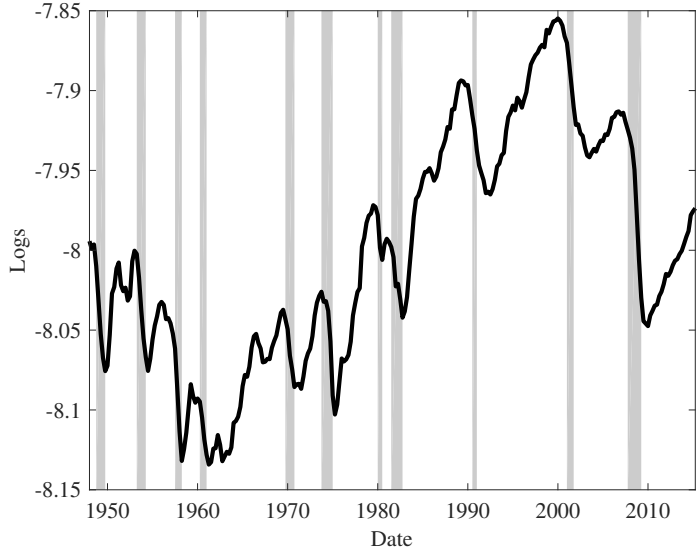
II.1. Stationary Variables

Cyclicality



II.1. Stationary Variables

Non-Farm Business (NFB) Hours Per Capita



II.1. Stationary Variables

Non Farm Business Hours per Capita Spectrum

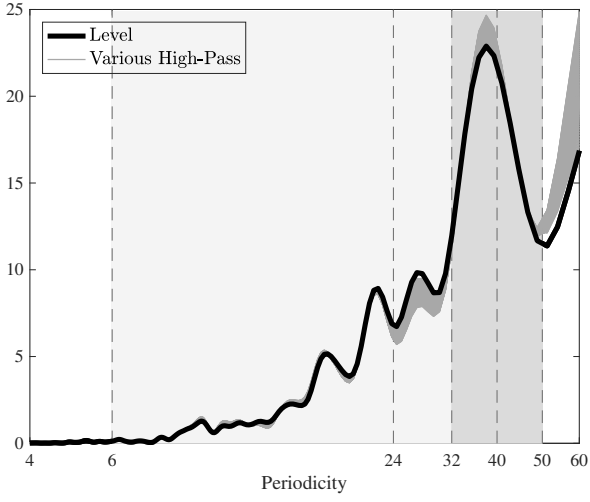
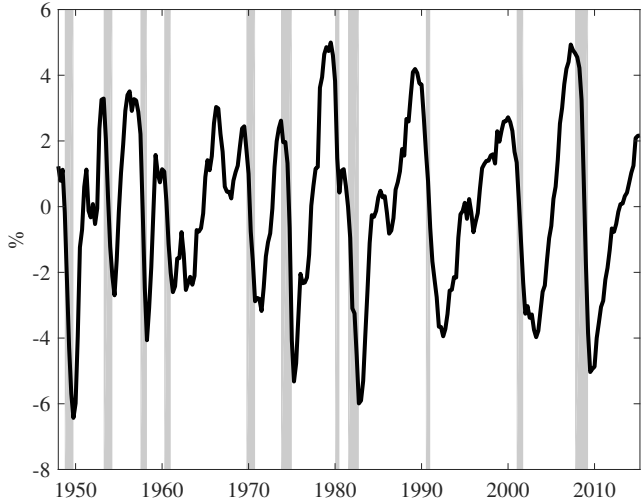
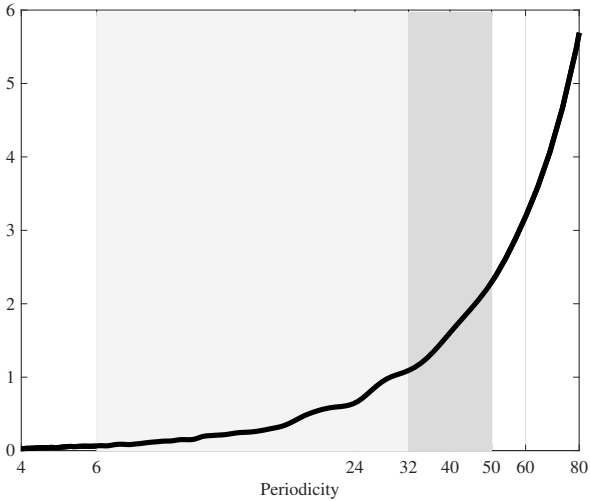


FIGURE 26 – Non Farm Business Hours per Capita, Highpass (50) Filtered



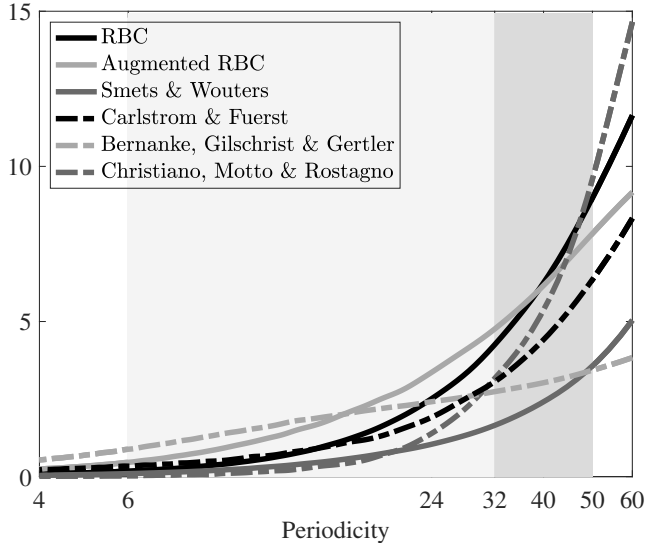
II.1. Stationary Variables

Hours Spectrum in Smets & Wouters' Model



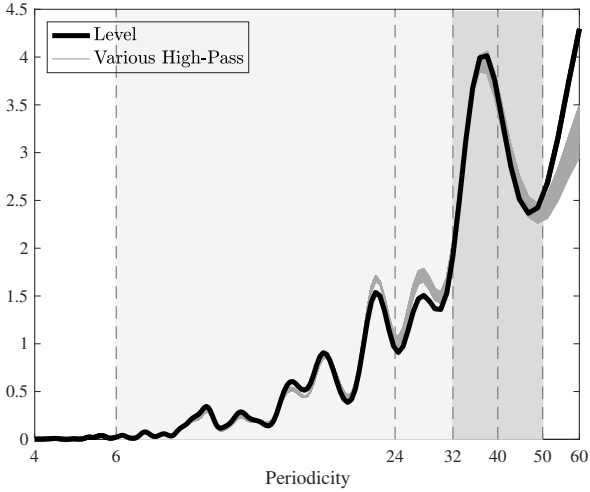
II.1. Stationary Variables

Hours Spectrum in Various Models



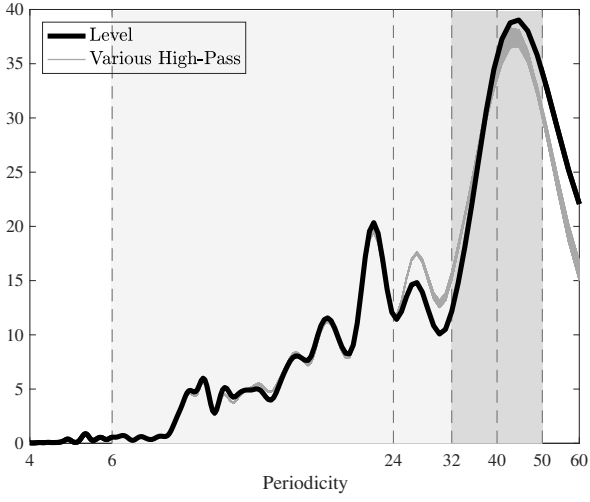
II.1. Stationary Variables

Unemployment Rate Spectrum



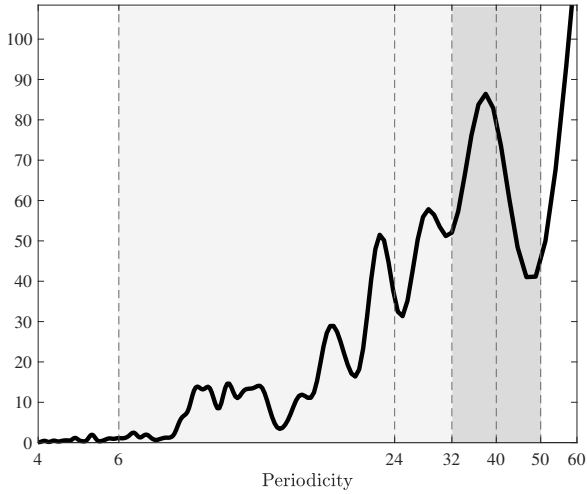
II.1. Stationary Variables

Capacity Utilization Spectrum



II.1. Stationary Variables

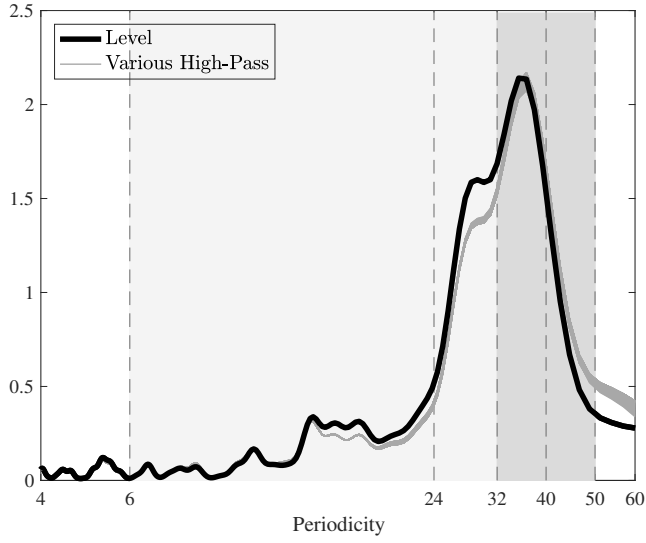
Investment-Output ratio



- The cycle is also a financial cycle
- (looking again at non-trending variables)

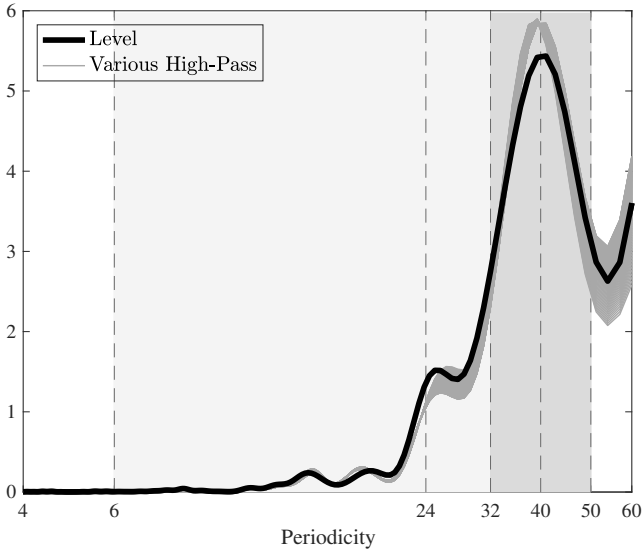
II.1. Stationary Variables

Chicago Fed National Financial Conditions Index



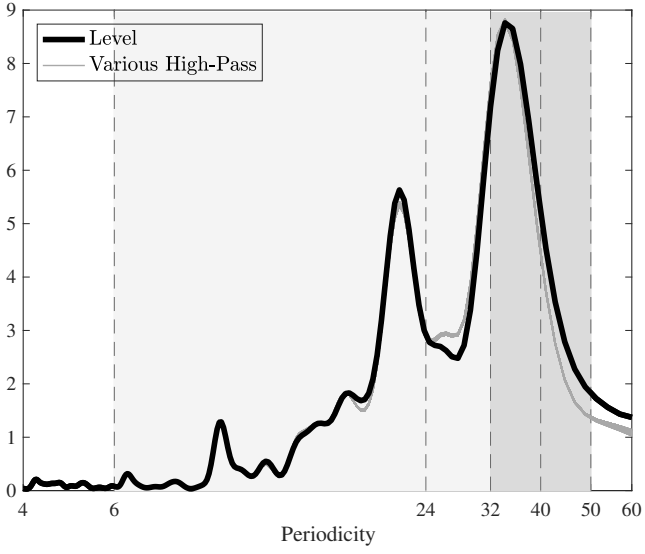
II.1. Stationary Variables

Delinquency Rate



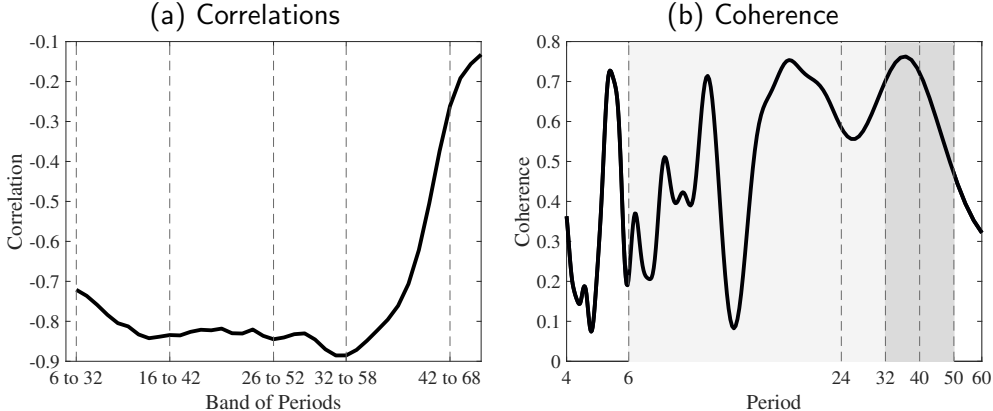
II.1. Stationary Variables

Spread (BBA bonds-FFR)



II.1. Stationary Variables

Correlations and Coherence Between (Filtered for Correlations, Level for Coherence) Hours and BAA/FFR Spread

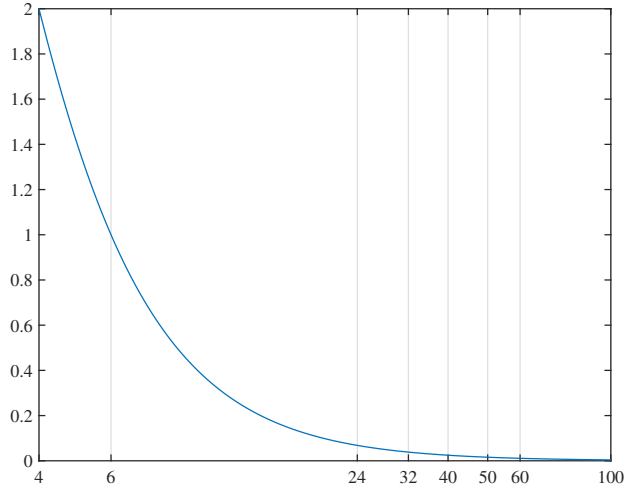


Panel (a) of This Figure displays the correlation between filtered Hours and BAA/FFR Spread, when the filter is a Band-Pass filter $(P, P + 26)$*, for* P *going from 6 quarters to 42. Panel (b) displays the coherence between the two series in levels. Sample is 1954Q3-2015Q2.*

- Traditional view of business cycles is a-cyclical :
  - × Spectral densities are thought monotonous
  - × This is what most models endogenously produce
  - × Exogenous forces do not look cyclical
- Data seems to tell us that there are indeed cycles
  - × because we have more observations
  - × because we look at non-trending variables (no need for stationarization)
- This is challenging for DSGE type models (that do not generate cycles)

- Want to look at many other series, but most are trending and appear non-stationary.
- How to treat non-stationary series. No easy answer.
- Assuming variable is sum of stationary business cycle component plus non-stationary I(1) component, can look at spectrum of first difference.
- However, the first difference filter has the effect of emphasizing high frequency movement of the stationary component, plus the non stationary component in laid on top. So may be hard to see property of the stationary component.
- If ready to assume that non-stationary part is RW, we can (often) re-construct the spectrum for the stationary component.
- Can also put through BP filter, example 2-100 or 2-200. But not clear what this gives you. Can work well under certain circumstances.

FIGURE 27 – Transfer of the First Difference Filter



- If

$$X_t = cyc_t + I(1)$$

- spectrum of first difference

$$S(\omega) = S_{\Delta cyc}(\omega) + S_{\Delta I(1)}(\omega)$$

- where $S_{\Delta cyc}(\omega)$ emphasizes high frequency movements, and $S_{\Delta I(1)}(\omega)$ may involve mass everywhere
- So may be difficult to infer property of the stationary component.
- In practice :
  1. take first difference,
  2. estimate the spectrum
  3. remove the zero frequency component
  4. invert the first difference filter to recuperate spectrum of the stationary part of the series

FIGURE 28 – Spectrum of first difference of hours worked

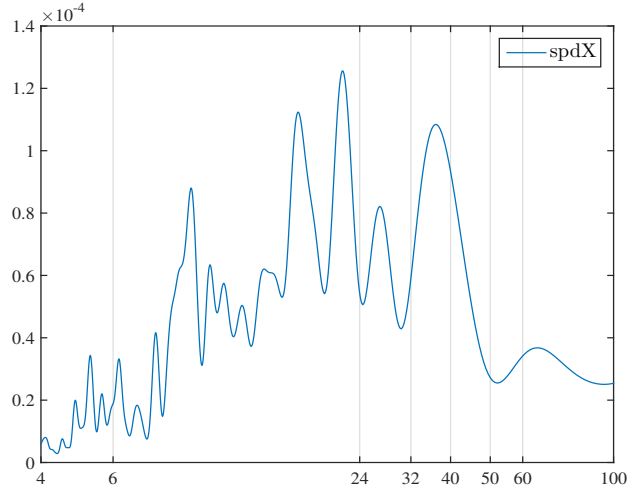
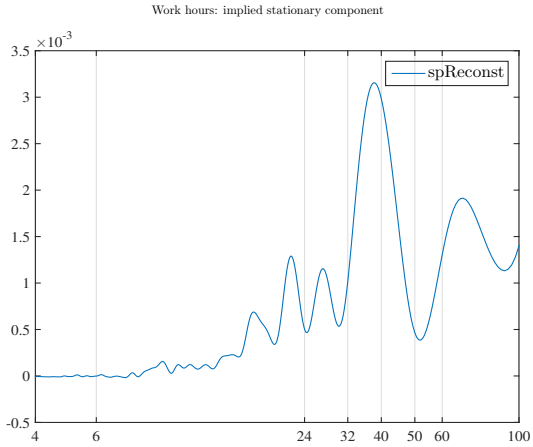


FIGURE 29 – Hours : RW Implied Component, BP filter



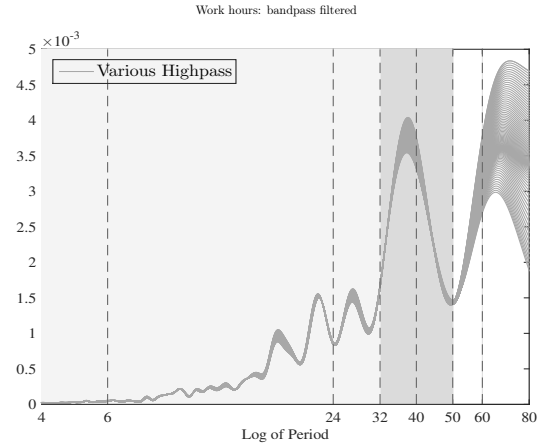


FIGURE 30 – Spectral Density, Level and High-Pass (2,100) to (2,200), Full Sample - Output

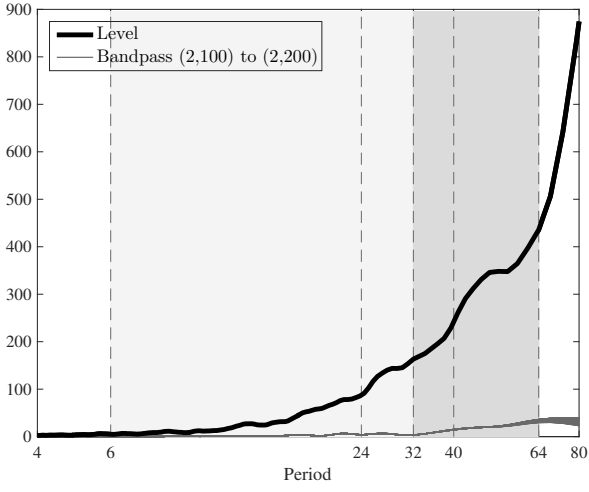


FIGURE 31 – Spectral Density, High-Pass (2,100) to (2,200), Full Sample - Output

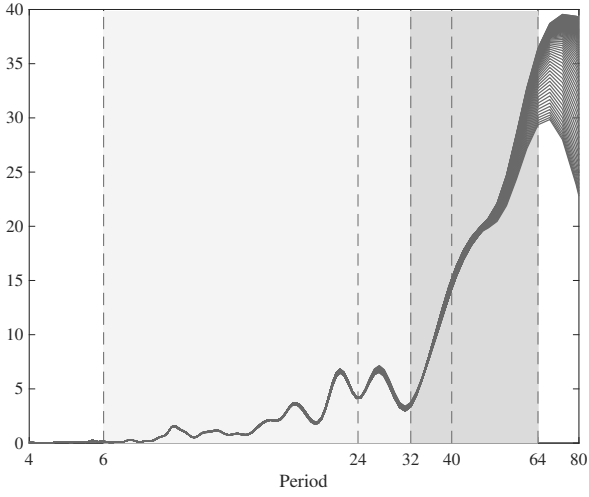


FIGURE 32 – GDP per capita : First difference

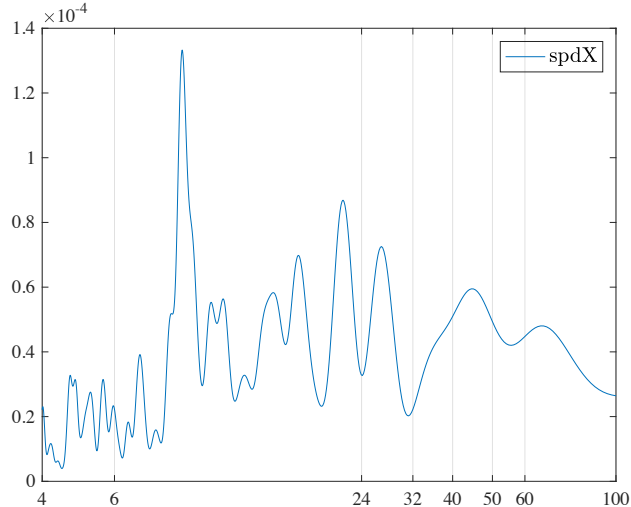
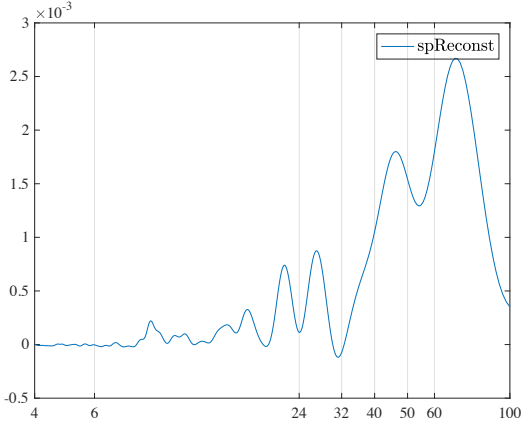
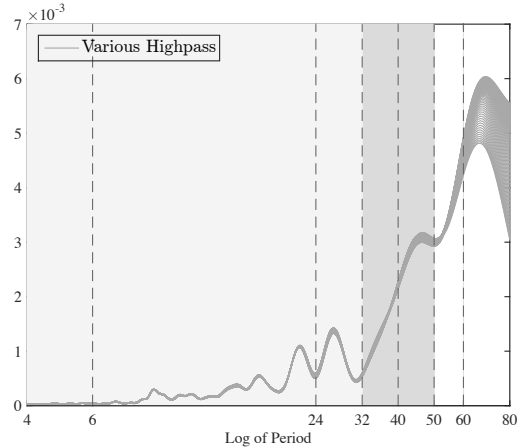


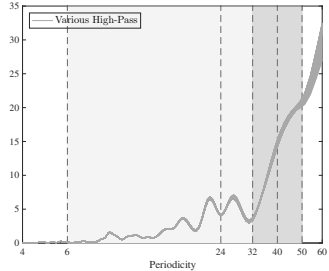
FIGURE 33 – Real GDP per capital : Implied Component and Various High-pass Filters



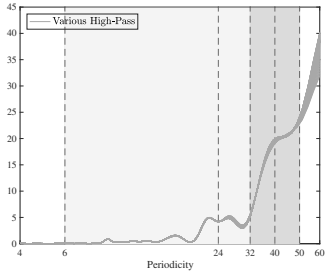


II.2. Non Stationary Variables

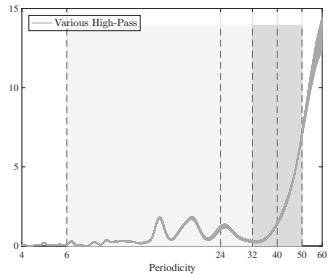
(a) GDP



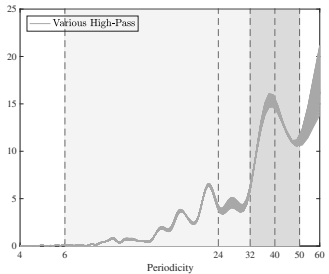
(b) hours+ TFP



(c) TFP



(d) hours



- Punchline : Not on a solid ground when looking at evidence of cycles with non stationary data.
- Assumptions on correlation between short and long run fluctuations impact the measurement.

Conclusion

- Suggest a need for theories which generate substantial endogenous recurrent cyclical behaviour, which may include real financial linkages.
  - × Kiyotaki-Moore (1997)
  - × Beaudry, Galizia and Portier (2016, 2017)
- Question : Why are these not pick up in current estimation approaches
  - × May not place enough weight on picking up this feature of the data.

Part I : Business Cycle Definition and Measurement

Part II : Revisiting the Evidence for Cycles

Part III : Generating Cycles through Dynamic Models with Interactions

- Here I present first an abstract dynamic model with strategic interactions
- These models have a macro flavour : a large number of atomistic agents
- I discuss the role of complementarities in generating multiplicity, multipliers, rich dynamics
- See my seminar later in the week for :
  - × I then present an extended NK model that features rich dynamics (up to limit cycles)
  - × I show that it can produce boom-bust cycles and peaks in spectral density.

- N players with payoffs function $V(e_j, e)$, where $e = \frac{1}{N}\sum_{j=1}^{N} e_j$
- $e \in [0, E]$
- $V_{11}(e_j, e) < 0$
- $e_j^\star(e)$ is the best response function, implicitly defined from

$$V_1(e_j, e) = 0;$$

- SCE is simply the planner solution

$$\max_e NV(e,e)$$

- FOC is

$$V_1(e,e) + V_2(e,e) = 0$$

- SOC is

$$V_{11}(e,e) + 2V_{12}(e,e) + V_{22}(e,e) < 0$$

- Assume for existence of an interior SCE

$$\lim_{e\to 0} V_1(e,e) + \lim_{e\to 0} V_2(e,e) > 0$$

$$\lim_{e\to E} V_1(e,e) + \lim_{e\to E} V_2(e,e) < 0$$

- *Positive spillovers* : $V_2(e_j, e) > 0$
- *Negative spillovers* : $V_2(e_j, e) < 0$
- *Strategic complementarity* : $V_{12}(e_j, e) > 0$
- *Strategic subtitutability* : $V_{12}(e_j, e) < 0$
- *Multiplier* :
  - × $\frac{\partial e_j^\star(e;\theta)}{\partial \theta} = \frac{V_{1\theta}}{V_{11}}$ is the partial equilibrium response of e to θ
  - × $\frac{\partial e}{\partial \theta} = \frac{V_{1\theta}}{V_{11}+V_{12}}$ is the general equilibrium response of e to θ
  - × There are multipliers when the general equilibrium response is in the same direction as the partial equilibrium one, and greater.
  - × An example of no multipliers would be the consumption of apples when taste for apple increases in a walrasian equilibrium.

- Agent aims at $\max_{e_j} V(e_j, e)$
- Agent j takes optimal action *given* the actions of others.
- Either she takes into account her effect on e or she does not
  - × If she does, FOC is

$$V_1(e_j, e) + V_2(e_j, e)\frac{\partial e}{\partial e_j} = V_1(e_j, e) + \frac{1}{N}V_2(e_j, e) = 0$$

  - × If she does not

$$V_1(e_j, e) = 0$$

  - × I focus on the second case (simple, but does not affect results
  - × N large $\rightsquigarrow$ close to the second case, which is the maximum departure from social optimum, as we will see

- Given by $V_1(e_j, e) = 0$
- Slope is (by full differentiation)

$$\rho = \frac{de_j}{de} = -\frac{V_{12}}{V_{11}}$$

- Best response can be increasing, decreasing, non monotonous, non linear,...

- SNE defined by

$$V_1(e, e) = 0$$

- to be compared to the SCE

$$V_1(e, e) + V_2(e, e) = 0$$

- Prop 1 : if there are spillovers, then the SNE is inefficient
- Prop 2 : If spillovers are positive, then the SNE levels of e are too low
- Prop 3 : Strategic complementarities are necessary and sufficient for multipliers
- Note : When there are positive spillovers and strategic compl., may want to refer to such equilibrium outcome as exhibiting *"deficient demand"*

- Prop 4 : There can be multiple equilibria. Strategic complementarities (and $-\frac{V_{12}}{V_{11}} > 1$ at some point) are necessary for multiplicity
- Prop 5 : If positive spillovers and multiple equilibria, then equilibria are Pareto ranked with higher actions being better.
- Corollary : If $e_j^\star(e) = e$ over an interval, and there are positive spillovers, then there is a continuum of equilibrium increasing in e.

- A Walrasian equilibrium can be seen as a special case of the above game situation.
- In this case, we can see agents as choosing actions – expenditures or supply decisions – taking prices as given, but where prices depend on aggregate actions
- What differentiates the Walrasian equilibrium is at an equilibrium $V_2 = 0$, ie, there are no spillovers.
- Therefore, the Walrasian equilibrium is efficient
- This is not because agents are small, but because the effects of ones actions - at the margin - does not have a positive nor negative effect on the others.

- This setup extends naturally to situation with imperfect information.
- For example,the θ in $V(e_j, e; \theta)$ may be a latent variable, which is only observed with error by agents.
- This is the global games framework.
- Some of the insights from the global games setup
  - × Imperfect information may eliminate multiple equilibria.
  - × Imperfect information affects the strength of complementarities (now reaction is a function of the expectation of the others actions, which depends on one's expectations of others expectation). This can cause expectation to play the role of a state variable
  - × This may allow for shocks to higher order beliefs (an example is Angeletos & La'O (2013))

III.2. A framework for Dynamics with Strategic Interactions

Environment

- N players
- Each agent accumulates E_i by playing e_i
- Decision rule and law of motion for E are

$$E_{it+1} = (1-\delta)E_{it} + e_{it} \tag{10}$$

$$e_{it} = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F(e_t) \tag{11}$$

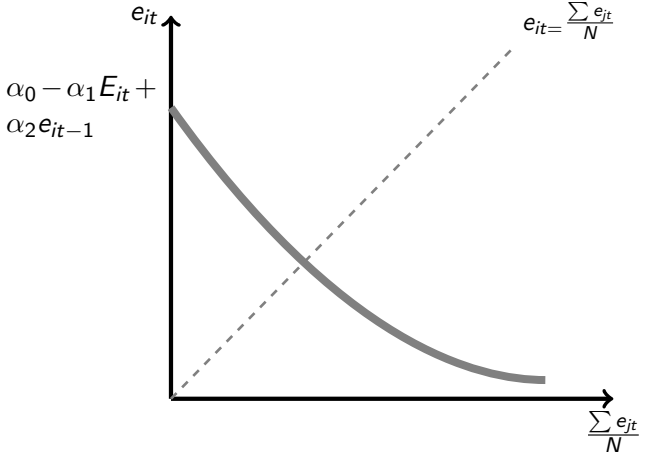
  with $e_t = \frac{\sum e_{jt}}{N}$
- all parameters are positive, $\delta < 1$, $\alpha_1 < 1$, $\alpha_2 < 1$.
- $\alpha_1 E_{it}$: Optimal size argument
- $\alpha_2 e_{it-1}$: Adjustment cost argument
- $F(e_t)$: Strategic Interactions
  - $F' > 0$: strategic complementarities
  - $F' < 0$: strategic substitutabilities

$$\begin{aligned} E_{it+1} &= (1-\delta)E_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F(e_t) \end{aligned}$$

- Define (E^s, e^s) as the steady state of the linear model where $F(\cdot) = 0$
- Normalize $F(e^s) = 0$

III.2. A framework for Dynamics with Strategic Interactions

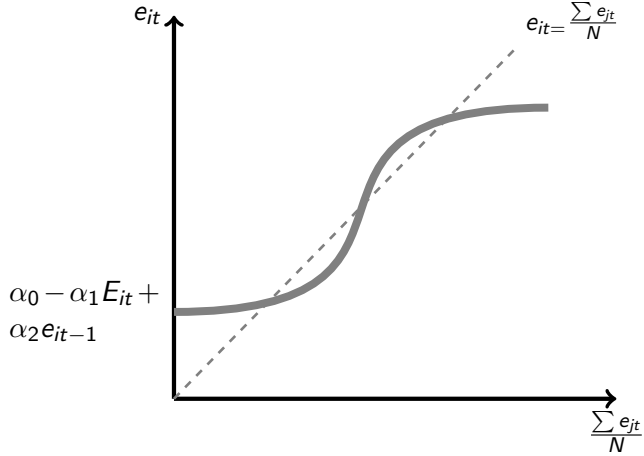
"Best response" rule for a given history - Strategic substitutability



$$e_{it} = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

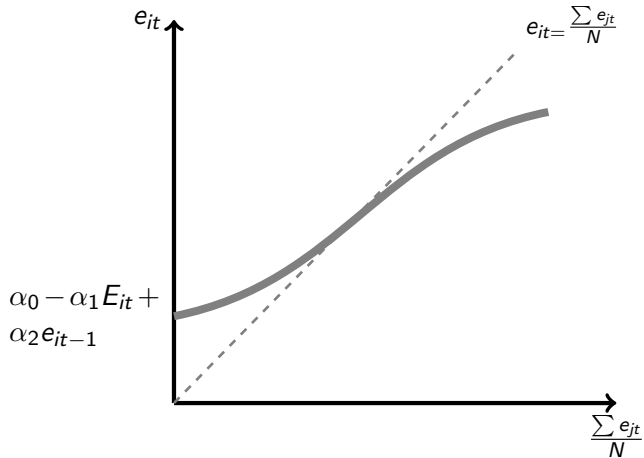
III.2. A framework for Dynamics with Strategic Interactions

"Best response" rule for a given history - Multiple Equilibria



$$e_{it} = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

- We restrict to the case where $F'(\cdot) < 1$, so that there are never multiple equlibria *within* period t



- Assume $\alpha_2 = 0$
- Then the model is

$$\begin{aligned} E_{it+1} &= (1-\delta)E_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 E_{it} + F(e_t) \end{aligned}$$

- It boils down to one order one dynamic equation in E_{it}

$$E_{it+1} = \alpha_0 + (1-\delta-\alpha_1)E_{it} + F(E_{t+1} - (1-\delta)E_t)$$

- So that, for symmetrical allocations

$$E_{t+1} = \alpha_0 + (1-\delta-\alpha_1)E_t + F(E_{t+1} - (1-\delta)E_t)$$

- Assume no strategic interactions : $F(\cdot) \equiv 0$

$$E_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)E_t$$

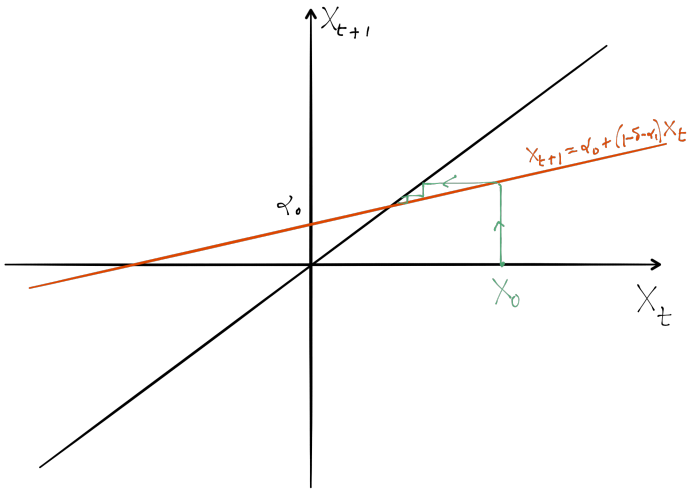
- Note that $(1 - \delta - \alpha_1)$ can be either > 0 or < 0.

Proposition 1

The Economy is globally stable as $|1 - \delta - \alpha_1| < 1$

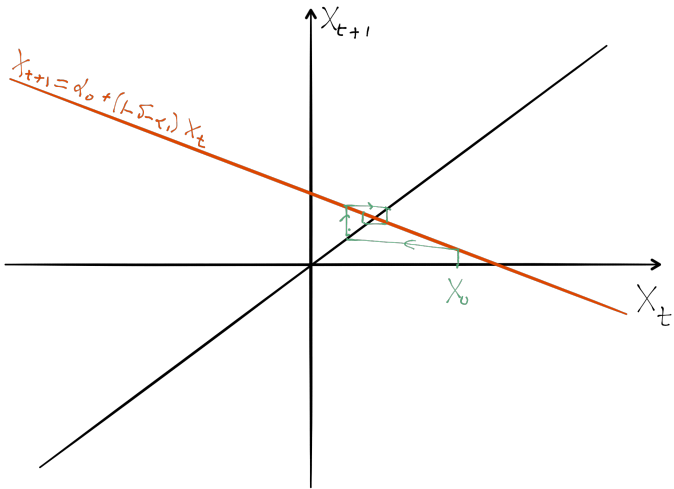
III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case



III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case



- Reintroduce strategic interactions

$$E_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)E_t + F(E_{t+1} - (1 - \delta)E_t)$$

- Linearize around the steady state E^s

$$E_{t+1} = (\alpha_0 + \delta E^s) + (1 - \delta - \alpha_1)E_t + F'(E^s)(E_{t+1} - (1 - \delta)E_t)$$

- so that

$$E_{t+1} = \frac{\alpha_0 + \delta E^s}{1 - F'(E^s)} + \frac{((1 - \delta)(1 - F'(E^s)) - \alpha_1)}{1 - F'(E^s)} \quad E_t$$

$$E_{t+1} = \frac{\alpha_0 + \delta E^s}{1 - F'(E^s)} + \frac{((1-\delta)(1 - F'(E^s)) - \alpha_1)}{1 - F'(E^s)} \quad E_t$$

Proposition 2

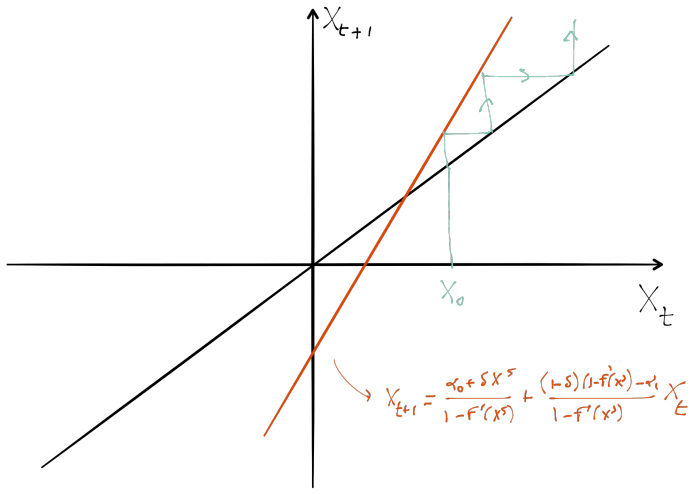
With strategic substitutability ($F'(\cdot) < 0$), the economy is always stable

Proposition 3

With strategic complementarity ($F'(\cdot) > 0$), there is always a level of complementarity smaller than 1 such that the SS is (locally) unstable.

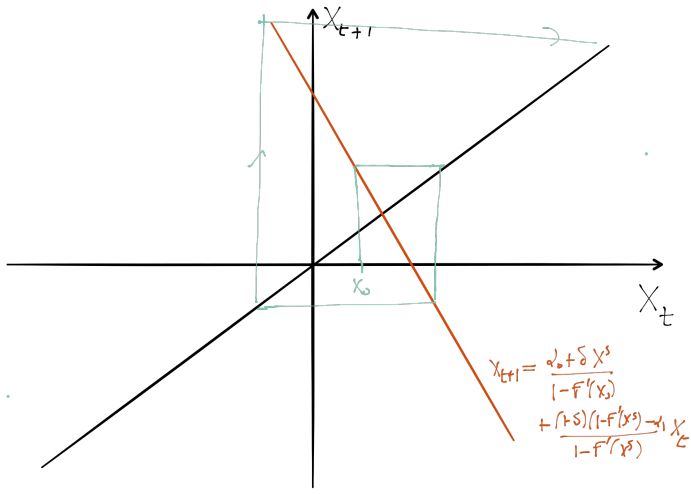
III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case

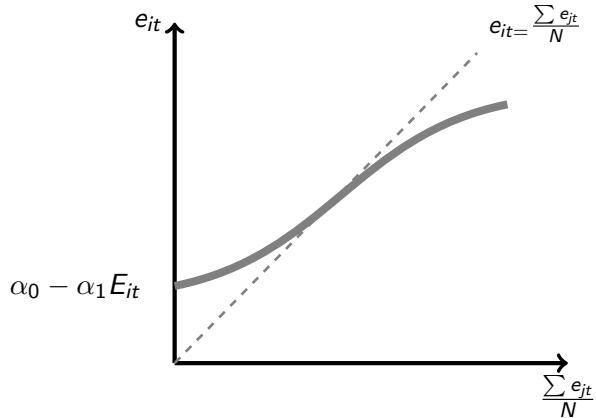


III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case



- The SS E^s becomes locally unstable
- "Real data" does not look explosive
- Let's assume that strategic complementarities dies out when the economy is far above of below E^s
- F is "S-shaped"



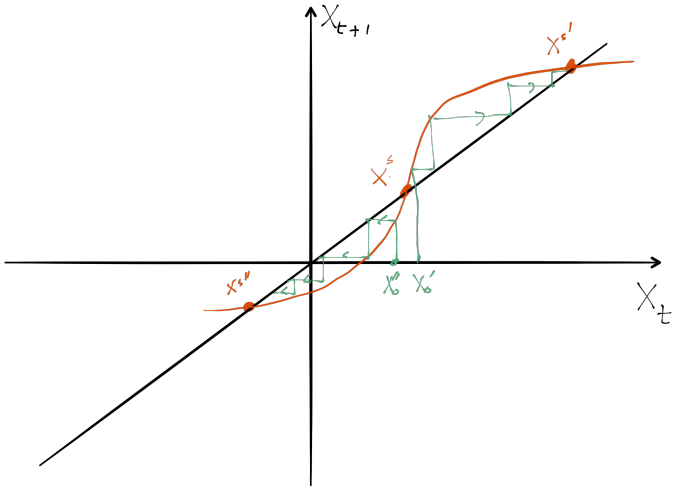
- Assume for simplicity that F is symmetric wrt E^s

$$E_{t+1} = \frac{\alpha_0 + \delta E^s}{1 - F'(E^s)} + \frac{((1-\delta)(1-F'(E^s)) - \alpha_1)}{1 - F'(E^s)} \quad E_t$$

- The dynamics of the economy will very much depend on whether $\frac{((1-\delta)(1-F'(E^s))-\alpha_1)}{1-F'(E^s)}$ is positive or negative
- Once non-linearities in F are taken into account,
  - × Positive : Hysteresis
  - × Negative : Limit cycle
- Need to think of environments that makes this slope is positive or negative

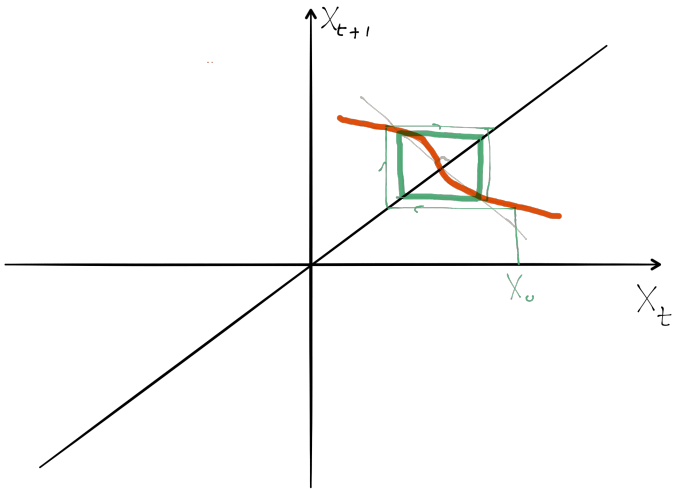
III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case



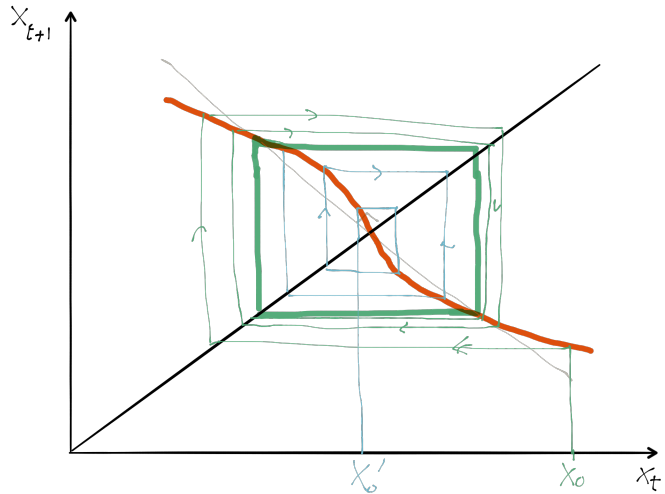
III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case



III.2. A framework for Dynamics with Strategic Interactions

One-Dimension Case



- Problem with one dimension model : the dynamics does not look like a business cycle
- E_t is *not* positively correlated.
- This needs not to be true in the more general model where $\alpha_2 > 0$

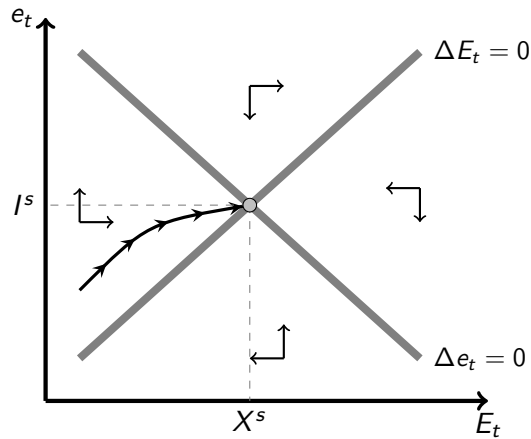
$$\begin{aligned} E_{it+1} &= (1-\delta)E_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F(e_t) \end{aligned}$$

- Without strategic interactions ($F(\cdot) \equiv 0$)
- When all agent behave symmetrically :

$$\begin{pmatrix} e_t \\ E_t \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_2 - \alpha_1 & -\alpha_1(1-\delta) \\ 1 & 1-\delta \end{pmatrix}}_{M_L} \begin{pmatrix} e_{t-1} \\ E_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}$$

Proposition 4

Both eigenvalues of the matrix M_L lie within the unit circle. Therefore, the system is stable.



Here we have assumed that the two eigenvalues are real and positive.

- Math
- Dynamics is

$$\begin{pmatrix} e_t \\ E_t \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_2 - \alpha_1 & -\alpha_1(1-\delta) \\ 1 & 1-\delta \end{pmatrix}}_{M_L} \begin{pmatrix} e_{t-1} \\ E_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_0 + F(e_t) \\ 0 \end{pmatrix}$$

- Local dynamics is

$$\begin{pmatrix} e_t \\ E_t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\alpha_2-\alpha_1}{1-F'(I^s)} & -\frac{\alpha_1(1-\delta)}{1-F'(I^s)} \\ 1 & 1-\delta \end{pmatrix}}_{M} \begin{pmatrix} e_{t-1} \\ E_{t-1} \end{pmatrix} + \begin{pmatrix} \left(1 - \frac{\alpha_2-\alpha_1}{1-F'(I^s)}\right) I^s + \left(\frac{\alpha_1(1-\delta)}{1-F'(I^s)}\right) X^s \\ 0 \end{pmatrix}$$

- When $F'(I^s)$ varies from $-\infty$ to 1, eigenvalues of M vary

Proposition 5

As $F'(I^S)$ varies from 0 to $-\infty$, the eigenvalues of M always stay within the unit circle and therefore the system remains locally stable.

Proposition 6

As $F'(I^s)$ *varies from* 0 *towards* 1, *the dynamic system will become locally unstable.*

(bifurcation)

- Here we can have a limit cycle with persistence
- Consider the steady state (E^s, I^s)
- Strategic complementarities : centrifugal force that pushes away from the steady state when close to.
- Accumulated variable X : centripetal force that pushes towards the steady state when away from.
- The steady state locally unstable, but forces push the economy back to the steady state when it is further from it.
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable

FIGURE 34 – Stable Steady State

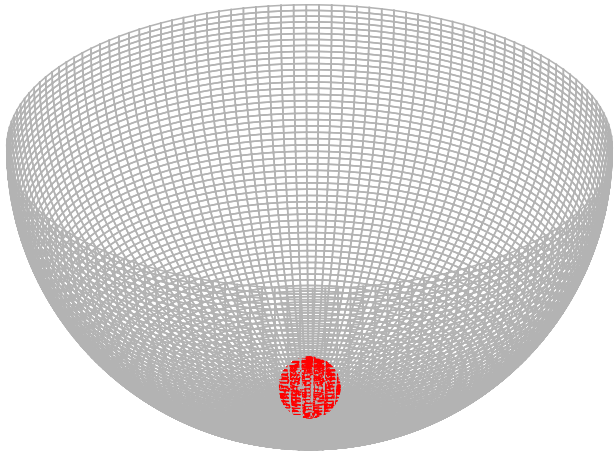


FIGURE 35 – HOPF Supercritical bifurcation : Attractive Limit Cycle

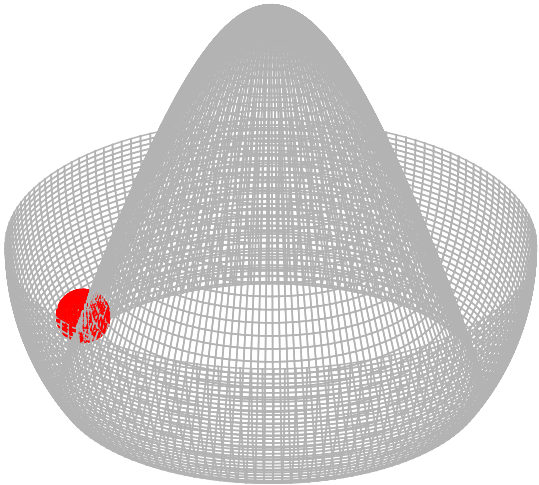


FIGURE 36 – Unstable Steady State

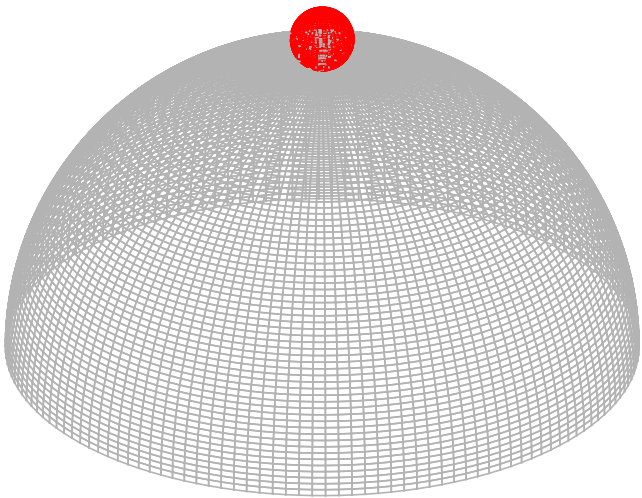
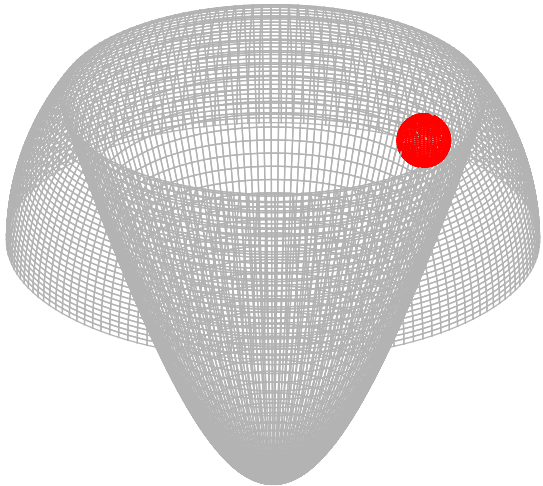
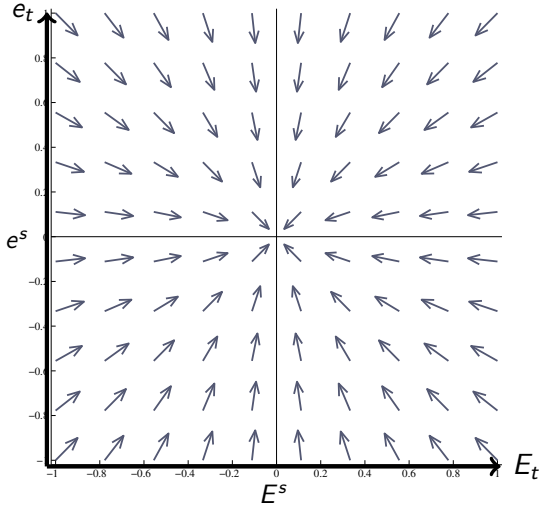


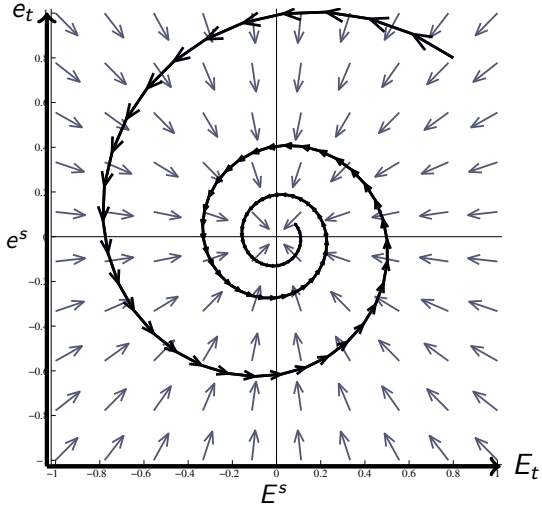
FIGURE 37 – HOPF Subcritical bifurcation : Repulsive Limit Cycle



III.2. A framework for Dynamics with Strategic Interactions

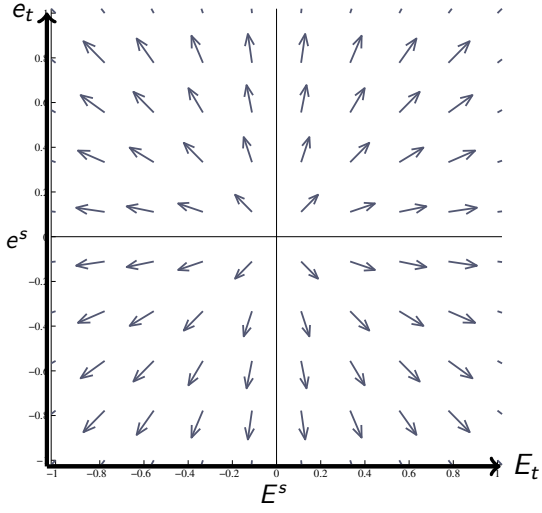
Two-Dimension Case - Global stability

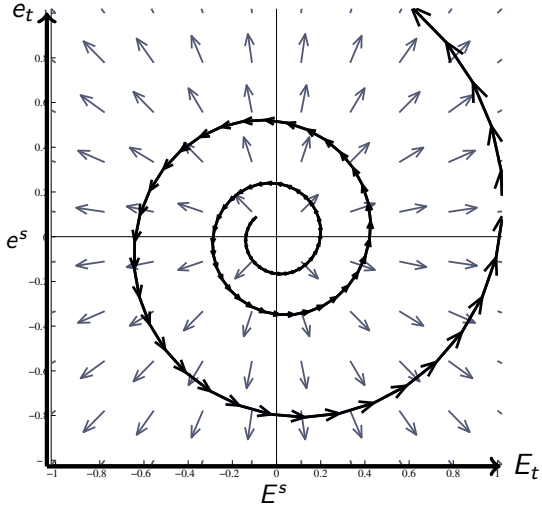



e_t
e^s
E_t
E^s
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

III.2. A framework for Dynamics with Strategic Interactions

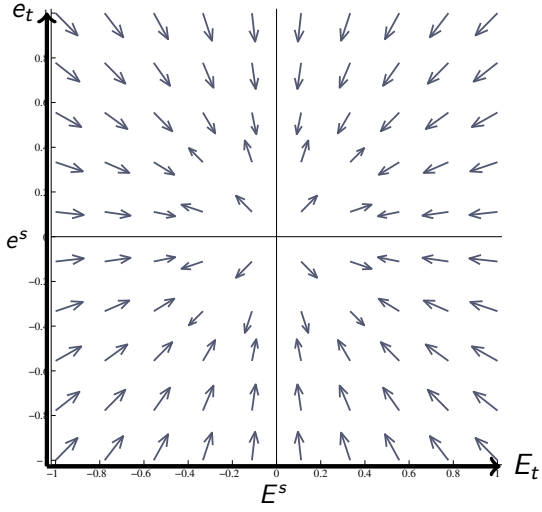
Two-Dimension Case - Global instability




e_t
e^s
E_t
E^s
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

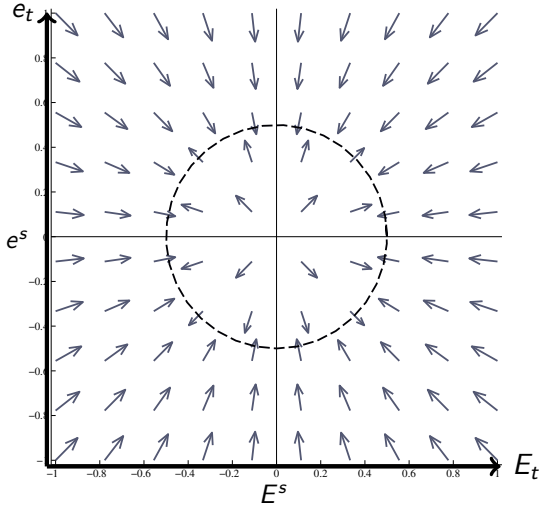
III.2. A framework for Dynamics with Strategic Interactions

Two-Dimension Case - Stable limit cycle



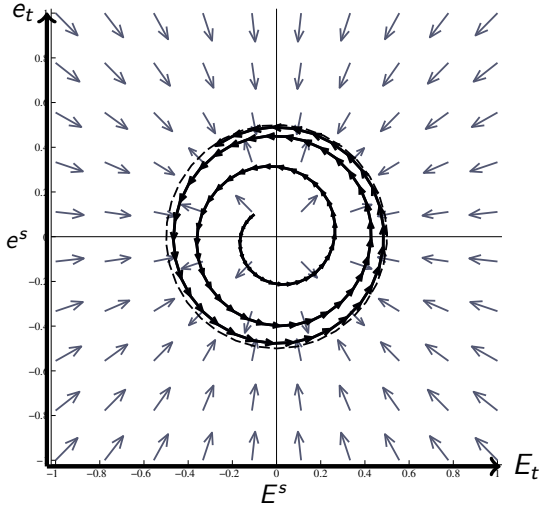
III.2. A framework for Dynamics with Strategic Interactions

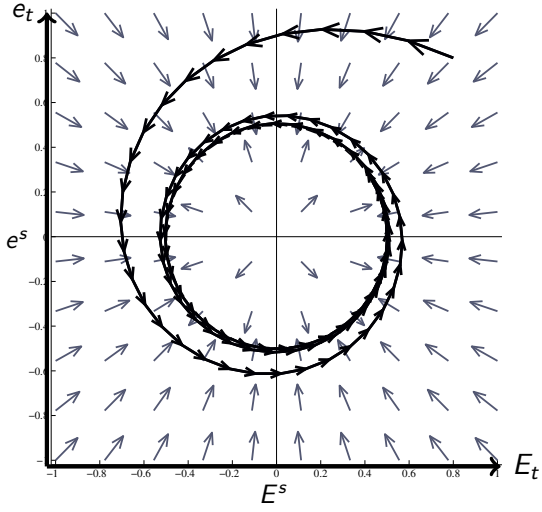
Two-Dimension Case - Stable limit cycle



III.2. A framework for Dynamics with Strategic Interactions

Two-Dimension Case - Stable limit cycle




e_t
e^s
E_t
E^s
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

- Discrete time version of the HOPF theorem.
- Consider the 2-dimensional smooth system

$$Y_{t+1} = F(Y_t; \alpha)$$

- Assume : Steady State for $\alpha = 0$: $F(\vec{0}; 0) = 0$
- Assume : Jacobian of F at (0,0) has two eigenvalues $(1 + \alpha)e^{\pm i\theta_0}$
- That system has a bifurcation at $\alpha = 0$
- *i.e.* the Steady State changes stability with α
- Then, the system $Y_{t+1} = F(Y_t; \alpha)$ has a limit cycle near the origin.
- Note : other possibilities is that one eigenvalue is 1 or -1.

- Nice theorem : we simply have to look at the *linearized* dynamics to prove existence of a limit cycle
- The parameter α is here the degree of strategic complementarities at the steady state $F'(I^s)$
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable

- 3 types of bifurcation
  - × Fold bifurcation : appearance of an eigenvalue equal to 1,
  - × Flip bifurcation : appearance of an eigenvalue equal to -1
  - × HOPF bifurcation : appearance of two complex conjugate eigenvalues of modulus 1.
- We are interested in HOPF bifurcation because the limit cycle will be "persistent"

Proposition 7

As $F'(I^s)$ varies from 0 towards 1, the dynamic system will become unstable and :

- *If $\alpha_2 > \alpha_1/(2-\delta)^2$, then a* Hopf *(*Neimark-Sacker*) bifurcation will occur.*
- *If $\alpha_2 < \alpha_1/(2-\delta)^2$, then a flip bifurcation will occur.*

- In the case of flip and HOPF bifurcation, a limit cycle appears
- In case of HOPF, the cycle is "smooth"
- Condition for an HOPF bifurcation :

$$e_{it} = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

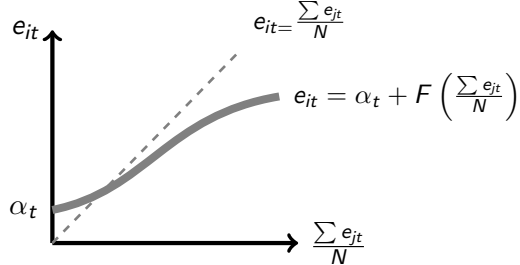
  - × as δ approaches 0, $\alpha_2 > 1/4$
  - × in general, what is needed is δ not too large and α_2 large enough

- In the case of the HOPF bifurcation, the limit cycle can be attractive (*the bifurcation is supercritical*) or repulsive (*the bifurcation is subcritical*)

Proposition 8

If $F'''(I^s)$ is sufficiency negative, then the HOPF *bifurcation will be supercritical. Therefore, the limit cycle is attractive.*

- $F''' < 0$ corresponds to an $S - shaped$ reaction function



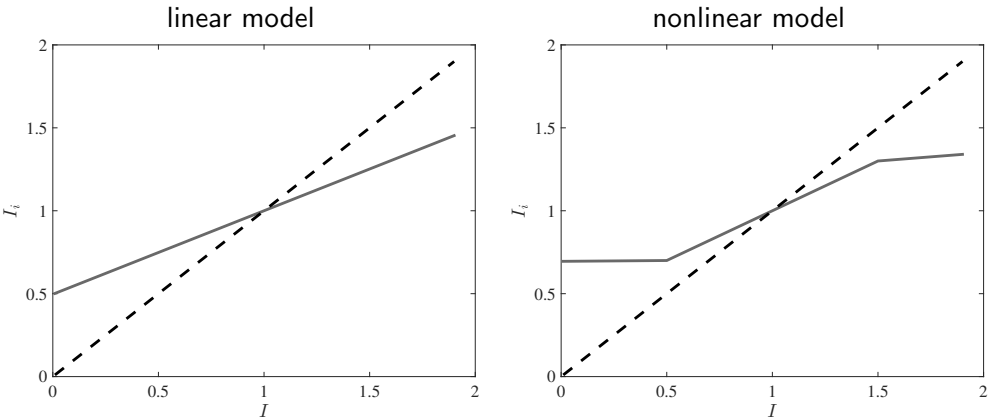
$$\alpha_t = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1}$$

III.6. Quantitative Reduced Form Limit Cycle Model

Model

- $E_{it+1} = (1-\delta)E_{it} + e_{it}$
- $e_{it} = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{it}^s}{N}, u_t\right)$
- $F(e_t, u_t) = \widetilde{F}(e_t) + u_t.$
- $u_t = \rho u_{t-1} + \varepsilon_t$
- $\widetilde{F}$: S-shaped and piecewise linear

FIGURE 38 – Best response rules in the numerical example



- Steady state is $e_{SS} = 1$, $E_{SS} = 10$
- Deterministic simulation : let $E_0 = 8$, $e_0 = 1$

FIGURE 39 – Deterministic simulation

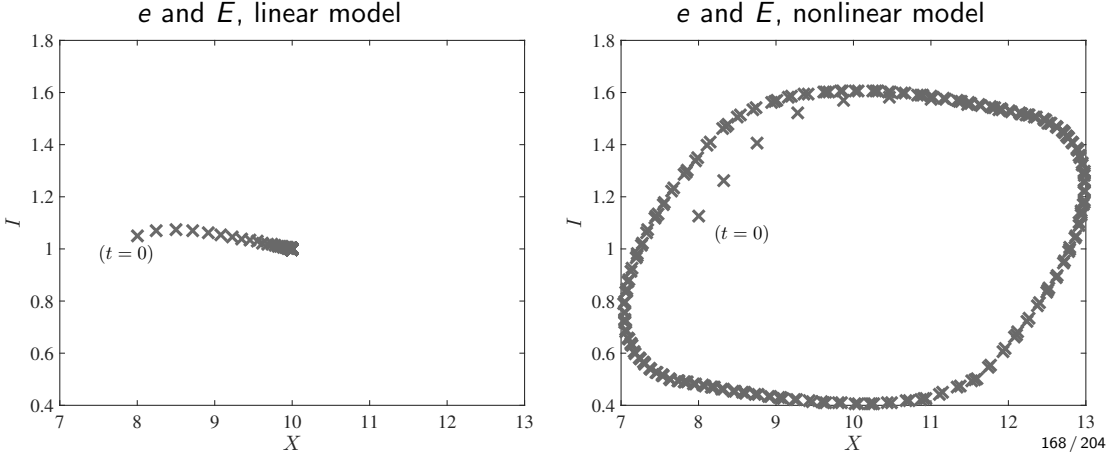


FIGURE 40 – Deterministic simulation

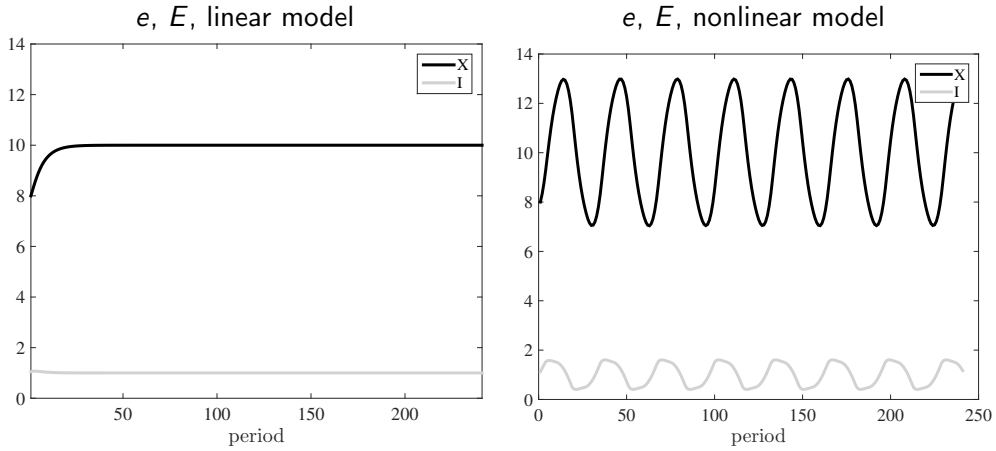


FIGURE 41 – Deterministic simulation of the nonlinear model

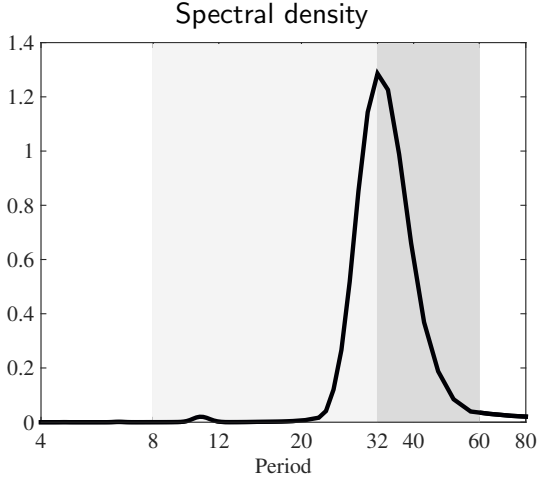


FIGURE 42 – One stochastic simulation

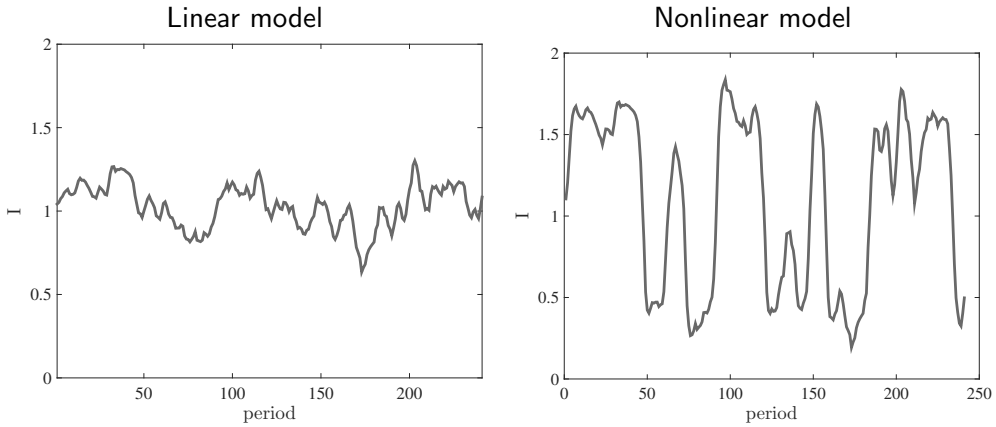
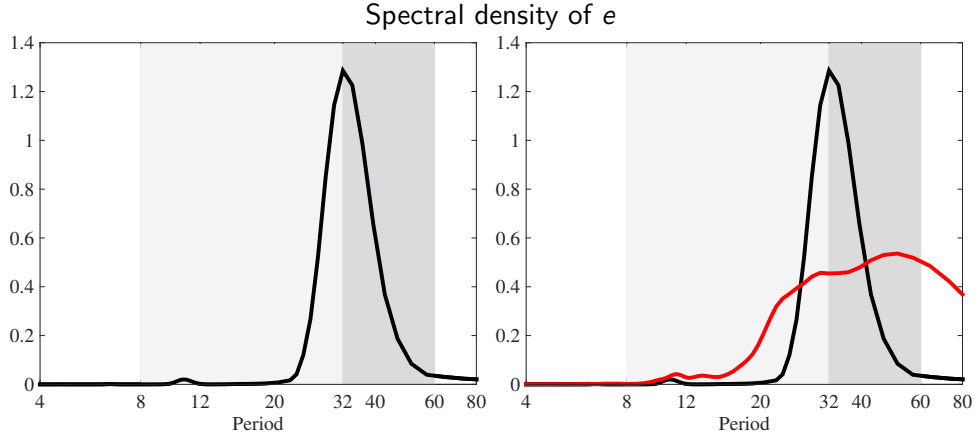


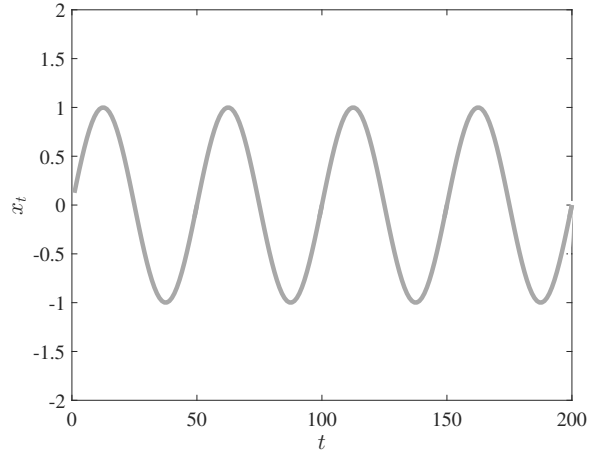
FIGURE 43 – Deterministic and Stochastic simulation of the nonlinear model



III.6. Quantitative Reduced Form Limit Cycle Model

What the results are not

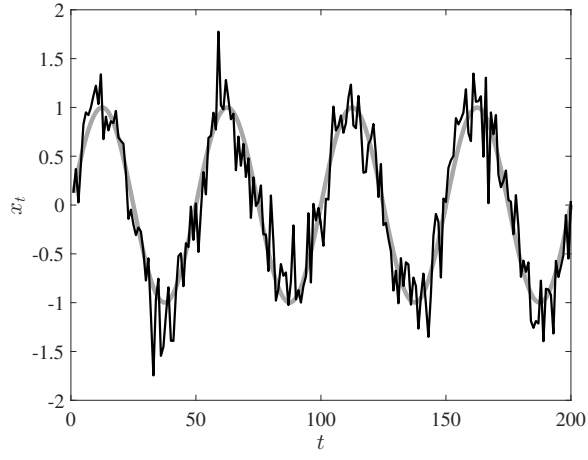
FIGURE 44 – $x_t = \sin(\omega t)$



III.6. Quantitative Reduced Form Limit Cycle Model

What the results are not

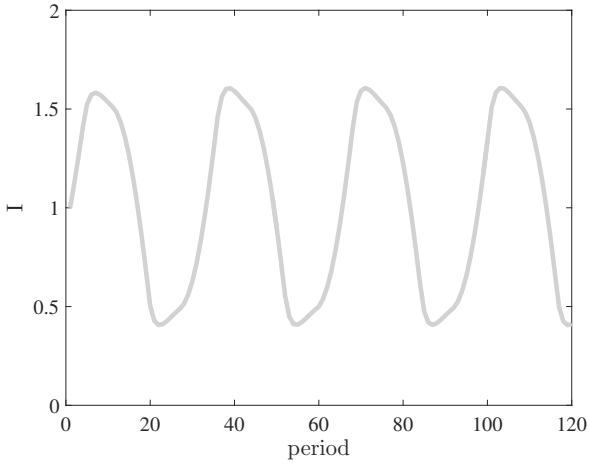
FIGURE 45 – $x_t = \sin(\omega t) + u_t$



III.6. Quantitative Reduced Form Limit Cycle Model

What the results are

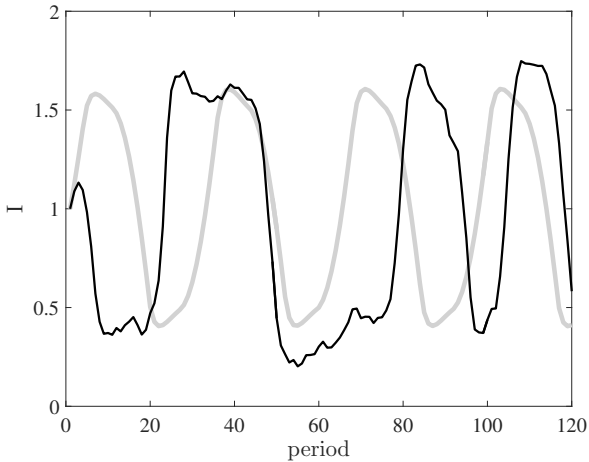
FIGURE 46 – Reduced Form Model



III.6. Quantitative Reduced Form Limit Cycle Model

What the results are

FIGURE 47 – Reduced Form Model



- $$e_{it} = \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + \alpha_3 E_t[e_{it+1}] + F\left(\frac{\sum e_{jt}}{N}\right)$$
- with accumulation remaining the same

$$E_{it} = (1-\delta)E_{it} + e_{it}$$

- Restrict attention to situations where this system is saddle path stable absent of complementarities.
- The local dynamics is described by the 3 eigenvalues of the linearized system

Proposition 9

If unique steady state, then no indeterminacy nor Fold bifurcations (always one positive and greater than one eigenvalue)

Proposition 10

If α_2 (sluggishness) sufficiently large, then the two other eigenvalues will become complex and

- *either will stay inside the unit disk (non locally explosive cycles)*
- *or will exit the unit disk (HOPF bifurcation and limit cycles)*

III.6. Adding forward looking elements

Set of results when one increases ρ

FIGURE 48 – Increasing complementarities ρ : local stability

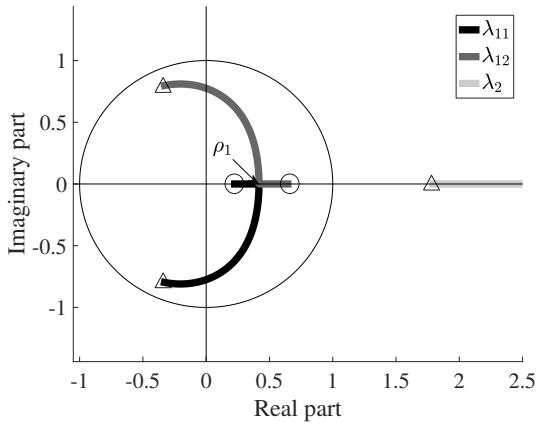


FIGURE 49 – Increasing complementarities ρ : local instability and limit cycle

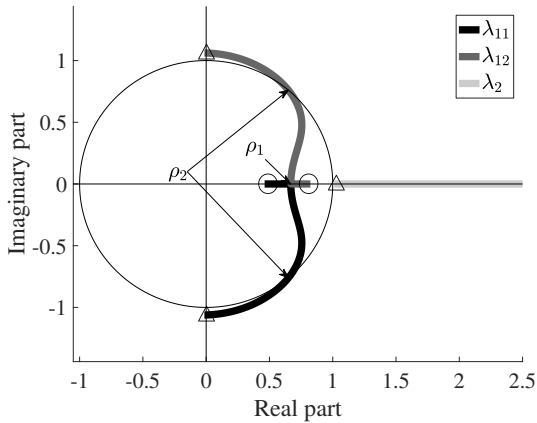


FIGURE 50 – A Saddle Limit Cycle

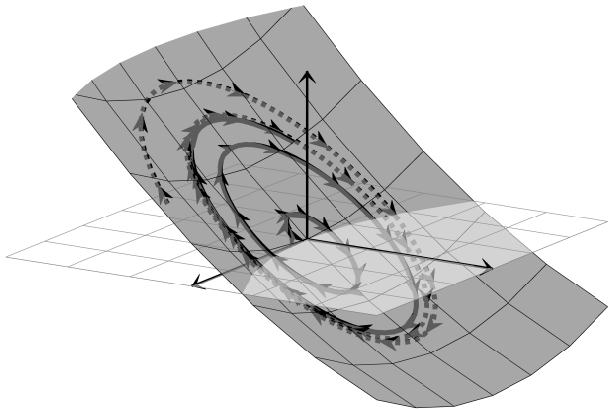


FIGURE 51 – Explosive path

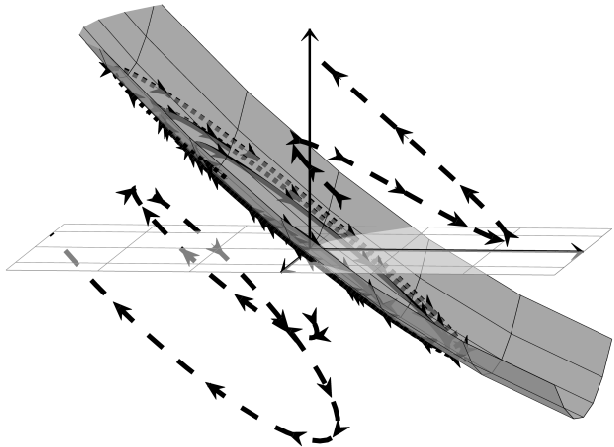
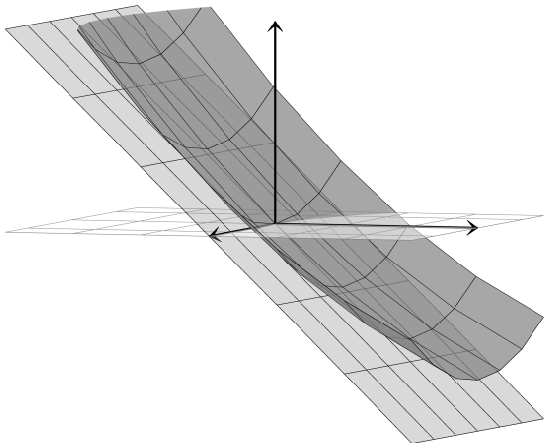


FIGURE 52 – Linear and non linear stable manifolds



- Weak complementarities + dynamics create cycles
- It can also create local instability and limit cycles
- One should be aware that estimating linear models is likely to biaise against this mechanism as one rules out local instability.

$$\begin{cases} e_{it} &= \alpha_0 - \alpha_1 E_{it} + \alpha_2 e_{it-1} + \alpha_3 E_t[a_{it+1}] + F\left(\frac{\sum e_{jt}}{N}\right) \\ E_{it} &= (1-\delta)E_{it} + e_{it} \end{cases}$$

- Stylized NK model which is extended to allow for the forces highlighted in our general structure.
- We add accumulation of durable-housing goods and habit persistence : *accumulation and sluggishness*
- Financial frictions : households face unemployment risk and cannot fully insure against it $\rightsquigarrow$ counter-cyclical risk premium $\rightsquigarrow$ *complementarities*
- Estimate parameters based on spectrum observations and higher moments. (use perturbation method and indirect inference)
- AR(1) discount factor shock, with persistence ρ

- The representation household who can buy/rent consumption services from the market.
- Households consumption services are supplied by firms and come from either the production of new goods or from the service flow of existing durables.

$$C_t = sX_t + F(L_t\theta_t)$$

where X_t is the stock of durable goods and $F(l_t\theta_t)$ is the current production of goods. The accumulation of durables is

$$X_{t+1} = (1-\delta)X_t + \psi F(L_t\theta_t)$$

where a fraction $\psi \leq (1-\delta)$ of newly produced goods become durables after one period.

- Households own the stock of durable-housing, rent it to firms, who supply back the consumption services as a composite good.

- Most likely, time is over
- I will discuss that in my seminar

- Labor indivisibility $\rightsquigarrow$ some workers are unemployed
- Spending decisions are made before labor market outcome is known
- The large family pays only for a fraction of its unemployed members.
- The interest rate which household face is then equal to the policy rate plus a risk premium

$$(1 + r_t) = (1 + i_t + r_t^p)$$

- The risk premium is a decreasing function of the economic activity, where the output gap and employment gap are interchangeable

$$r_t^p = G(\ell_t)$$

- The interest premium is generically a non-linear function of activity.
- The households Euler will have the familiar form (assuming external habit)

$$U'(C_t - \gamma C_{t-1}) = \beta_t E_t U(C_{t+1} - \gamma C_t)(1 + r_t)$$

- Firms face stochastic opportunities to change price (Calvo)
- Have Taylor rule of form

$$i_t = E_t \pi_{t+1} + \phi E_t \ell_{t+1}$$

where ℓ_t is (log) employment or output gap, and $U(\cdot) = \ln(\cdot)$

- Estimate parameters of model by SMM
- Targets
  - × spectrum of hours worked on the frequencies 2-50
  - × spectrum of interest rate spread on the frequencies 2-50
  - × Set of other higher moments (correlation, kurtosis and skewness of hours and spread)
- We estimate the model
  - × Without complementarities (No Risk Pemium)
  - × With complementarities, but linear approx.
  - × With complementarities, but order 3 approx.

FIGURE 53 – Spectral Density, Level and Bandpass (2,100) to (2,200), Full Sample - Non Farm Business Hours

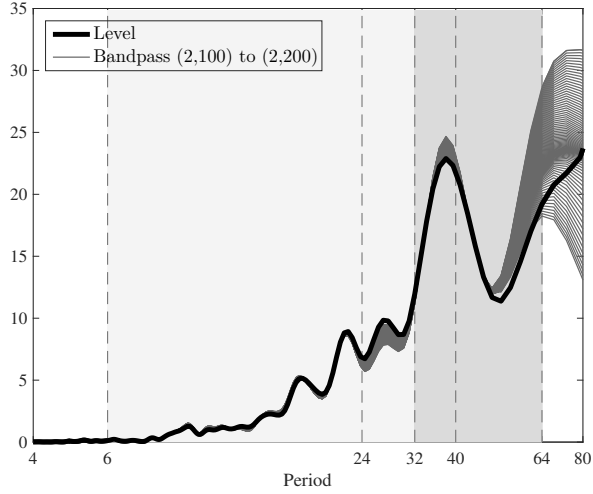


FIGURE 54 – Spectrum of Hours, No Risk Premium

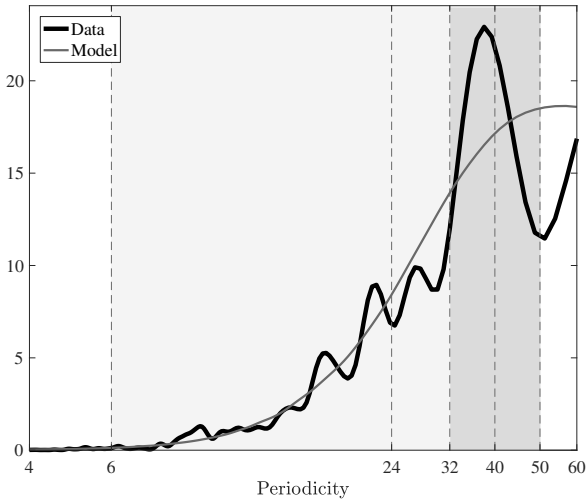


FIGURE 55 – Spectrum of Hours, Linear Risk Premium

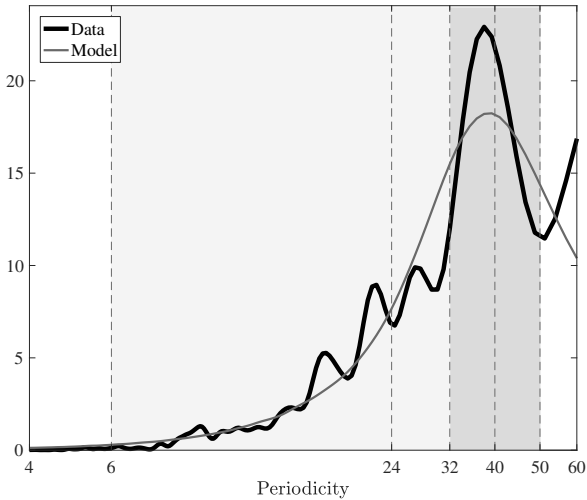


FIGURE 56 – Spectrum of Hours, Non-Lin. Risk Premium

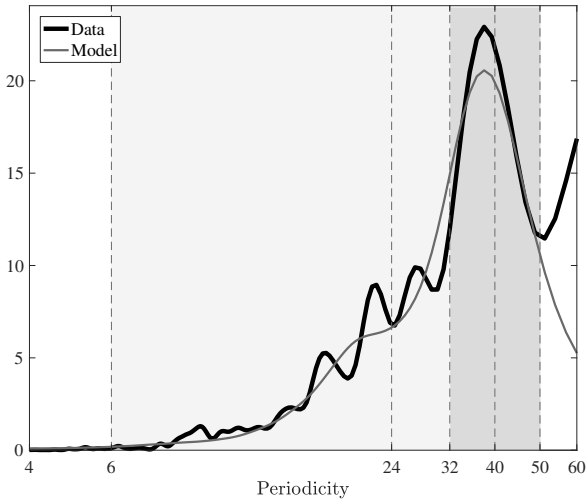


FIGURE 57 – Hours Spectrum in Smets & Wouters' Model

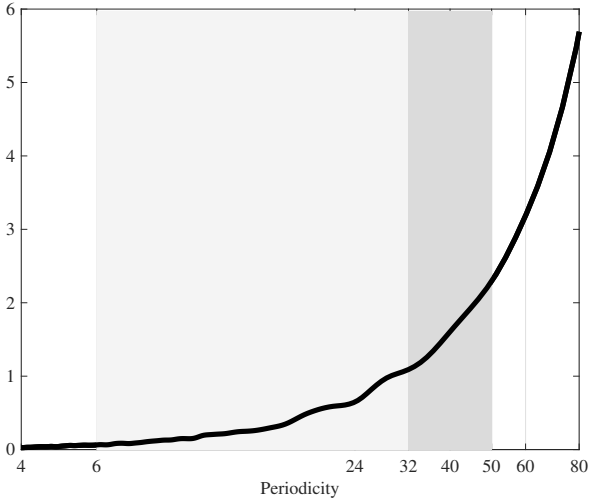


TABLE 2 – Parameter Estimates

		Non-Lin. RP	Lin. RP	No RP
ω	CRRA	0.2997	0.2408	0.2408*
γ	Habit	0.5335	0.5876	0.8346
ψ	Initial deprec.	0.4000	0.2994	0.1154
ϕ	Taylor rule	0.0421	0.0467	0.0000
R_2	Risk prem.	0.00017	—	—
R_3	Risk prem.	0.00006	—	—
$\bar{f}$	Debt backing	0.8668	0.8827	1*
b	Default cost	0.2870	0.3568	—
ρ	Autocorr.	-0.0000	0.1387	0.8346
σ	Innov. s.d.	0.0157	0.0272	0.0335
$s.d.(\mu)$	Uncond. s.d.	0.0157	0.0275	0.0609

TABLE 3 – Eigenvalues

	Non-Lin. RP	Lin. RP	No RP
λ_1, λ_2	$1.1032 \pm 0.2164i$	$0.9258 \pm 0.1372i$	0.8346, 0.8346
$\lvert\lambda_j\rvert$	1.1242	0.9359	0.8346

FIGURE 58 – Sample Draw for Hours

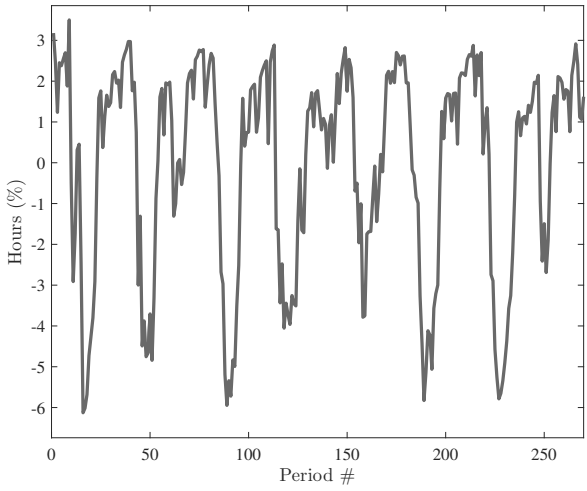


FIGURE 59 – Sample Draw for Hours, no shocks

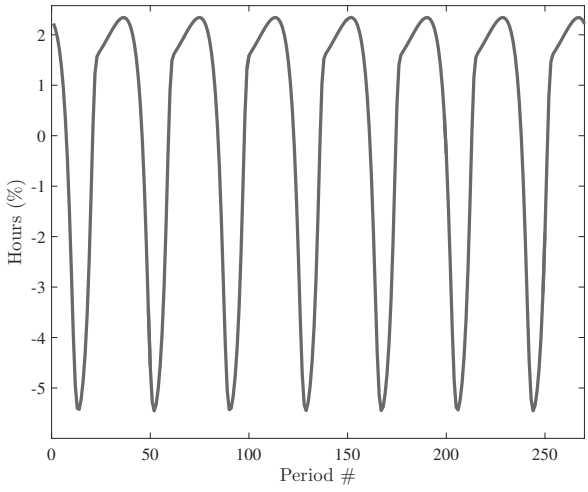
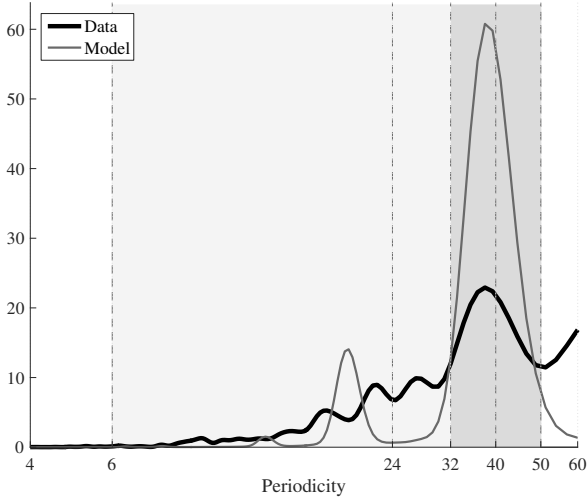


FIGURE 60 – Spectrum for Hours, no shocks



III.6. Results From Estimation

- Shocks are important for explaining the data.
- However, we estimate them to be i.i.d. in case with limit cycle.
  - × Almost all dynamics are internal.
  - × Main purpose : makes cycle *irregular*.
- Important : Adding shocks to a model with a limit cycle does not simply result in a predictable cycle with noise around it.
  - × Shocks have temporary amplitude effect, but "random-walk" effect on phase.
  - × i.e., permanently shift the whole cycle forward or backward in time.
  - × Phase of cycle completely unpredictable in the infinite future.

- Estimate the linear and non linear version over alternative frequencies : 2-32, 2-50, 2-60, 2-100.
- Main results
  - × over 2-32 : come back to case where the exogenous driving forces is the cycle
  - × Our results not changed much if go to 2-60 or 6-50, 6-60.. Almost all dynamics are internal
  - × However, if extend further, example 2-100, back to exogenous driving force dominating

III.6. Results From Estimation

Estimating over different frequency windows

FIGURE 61 – Estimation over 2-100

