Macroeconomic Fluctuations as Emergent Phenomena.

Instability Hub Meeting

Warwick, March 2019

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- There are many papers that highlight environments in which macro outcomes can depart substantially from agents desires.
- Even with "smoothness loving" forward looking agents, aggregate outcomes can take the form
  1. Indeterminacies and the emergence of animal spirits
  2. Hysteresis with very temporary shocks potentially having very long term effects
  3. Limit cycles and endogenous boom bust phenomena.
  4. Chaos
- Not clear to us what feature of the macroeconomy cause what.
- Our goal is to understand better what causes what.

Roadmap

1. Motivating Observations : Local Instability and Cycles
2. A Reduced Form (Agent Based) Approach
3. Structural Approach

Roadmap

1. Motivating Observations : Local Instability and Cycles
2. A Reduced Form (Agent Based) Approach
3. Structural Approach

- Let X_t be a cyclical measure of activity, example: hours worked or unemployment rate.
- Consider estimating an AR process (assuming zero mean).

$$X_t = A(L)X_{t-1} + \epsilon_t$$

  - × If roots of $I - A(L)$ are sufficiently outside of unit circle, we tend to take as evidence of stability.
  - × If very close to 1, we worry about a unit root.
- We often tend to disregard the possibility of roots inside the unit disc,
  - × as this would imply explosive behavior;
  - × because it is not found in the data

- But suppose instead the DGP process is of form

$$X_t = A(L)X_{t-1} + \alpha F(X_{t-1}) + \epsilon_t \qquad F(0) = F'(0) = 0$$

- where $F(\cdot)$ is a non-linear function and α may be very small.
- The stability of the zero steady state will still depend only on roots of $I - A(L)$.
- Hence, linear approx unchanged and dropping $\alpha F(X_{t-1})$ in estimation may seem reasonable.
- However, this could lead to substantial bias in estimation of roots of $A(L)$ if system is locally unstable.
- Consider AR(3) example with small cube term ($F(x) = x^3$).

- Assume the following DGP

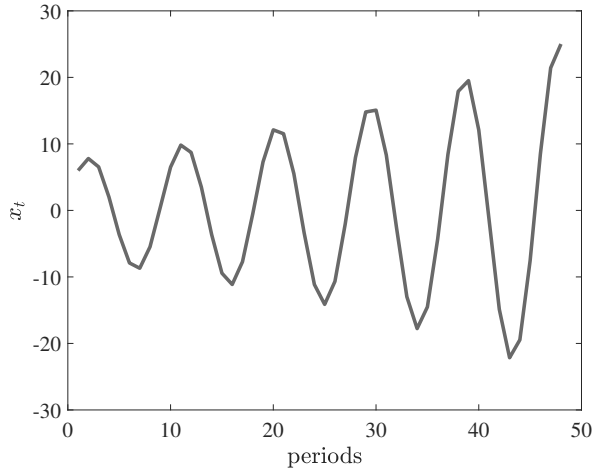
$$x_t = \alpha x_{t-1} - 0.6x_{t-2} - 0.3x_{t-3} - 0.01x_{t-1}^3 + .25\epsilon_t, \tag{1}$$

- α takes values in $[0.5, 1.5]$
- $\rightsquigarrow |\lambda|_{\max} \in [0.94, 1.4]$.
- Let's look at the case in which $|\lambda|_{\max} = 1.026$.

1. Motivating Observations : Local Instability and Cycles

Introductory example

Figure 1: Theoretical Impulse Response when ν is set to 0

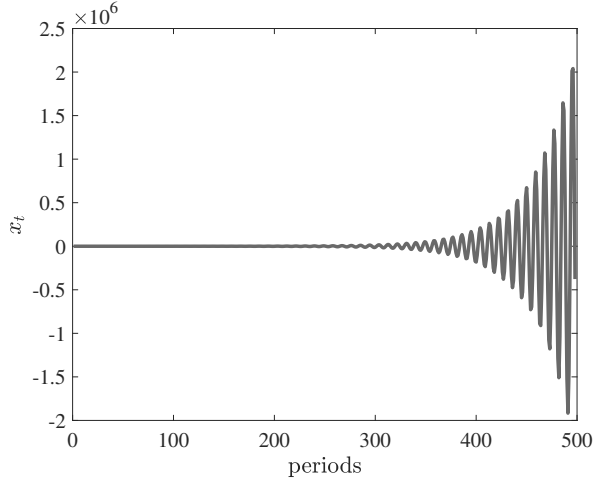


$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

1. Motivating Observations : Local Instability and Cycles

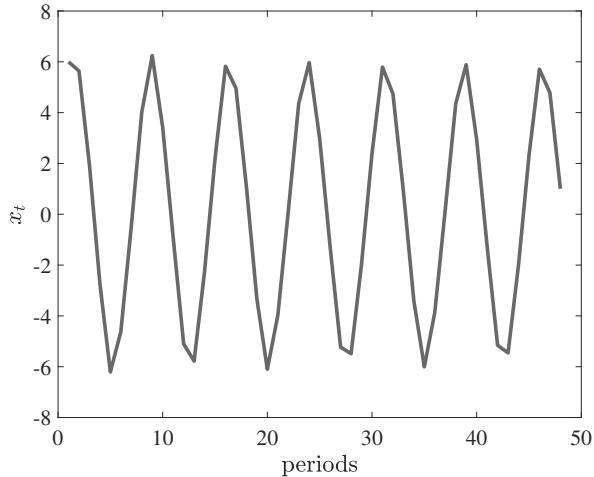
Introductory example

Figure 2: Theoretical Impulse Response when ν is set to 0



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

Figure 3: Theoretical Impulse Response in the Nonlinear Model

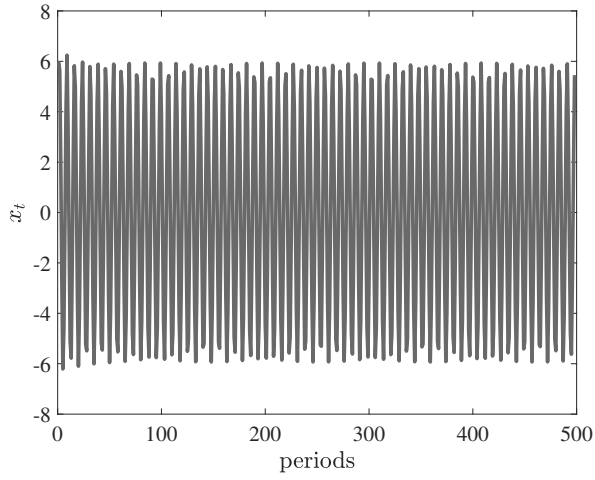


$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

1. Motivating Observations : Local Instability and Cycles

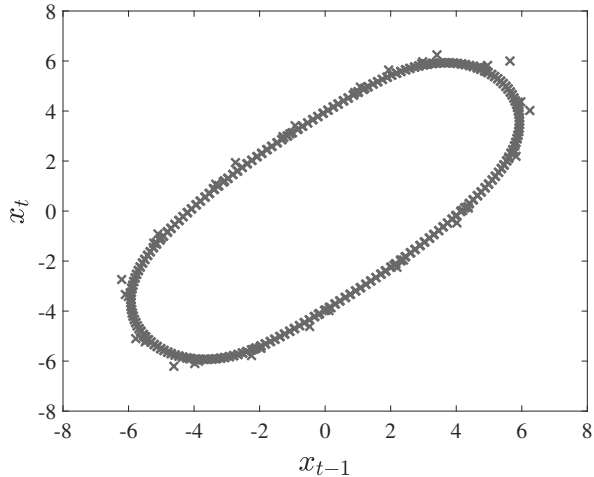
Introductory example

Figure 4: Theoretical Impulse Response in the Nonlinear Model



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

Figure 5: The Limit Cycle in the Nonlinear Model



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

$$x_t = \alpha x_{t-1} - 0.6x_{t-2} - 0.3x_{t-3} - 0.01x_{t-1}^3 + .25\epsilon_t, \tag{2}$$

- Take many α in $[0.5, 1.5]$
- $\rightsquigarrow |\lambda|_{\max} \in [0.94, 1.4]$.
- Simulated the DGP (2) for each α and estimate the two specifications:

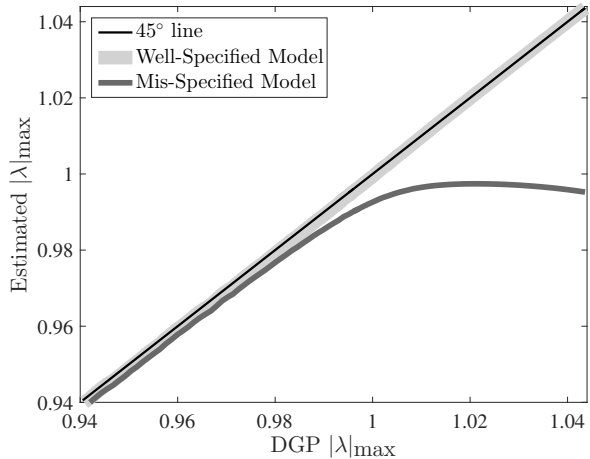
$$\begin{aligned} x_t &= \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} \quad + \nu x_{t-1}^3 \quad + \epsilon_t \\ x_t &= \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} \qquad\qquad + \epsilon_t \end{aligned}$$

- Compare estimated $|\lambda|_{\max}$ for well- and mis-specified models with the true $|\lambda|_{\max}$.

1. Motivating Observations : Local Instability and Cycles

Introductory example

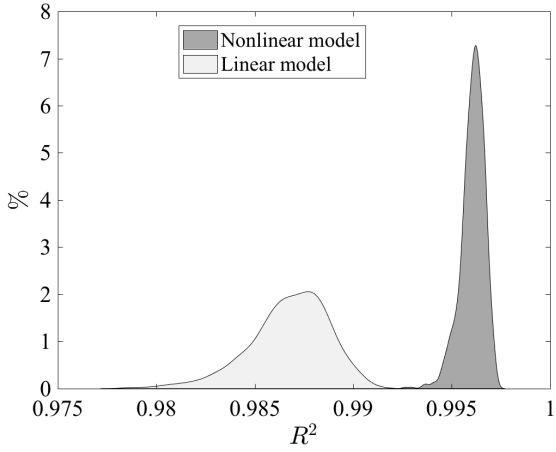
Figure 6: $|\lambda|_{\max}$ for the Linear and Nonlinear estimation when the DGP is Nonlinear



1. Motivating Observations : Local Instability and Cycles

Introductory example

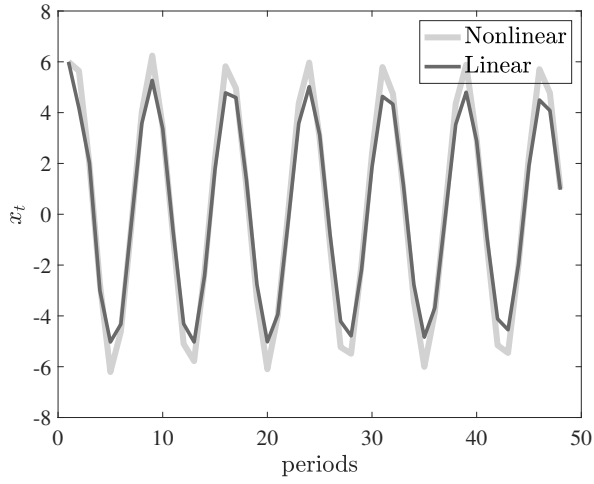
Figure 7: R^2 for the Linear and Nonlinear estimation when true $|\lambda|_{\max} = 1.02$ (1000 simulations, 300 observations per simulation)



1. Motivating Observations : Local Instability and Cycles

Introductory example

Figure 8: Estimated Nonlinear and Linear Model Impulse Response

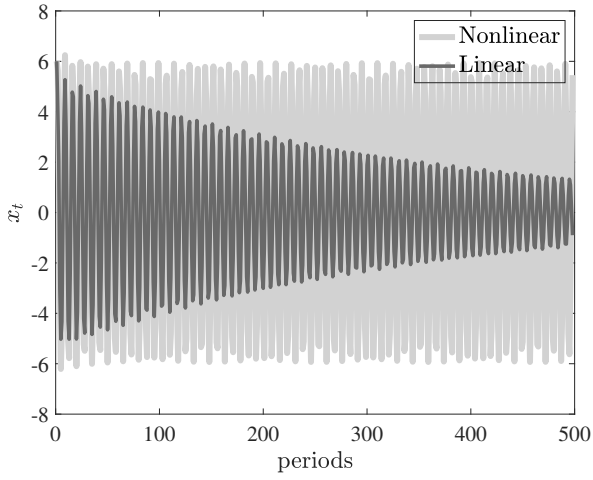


$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

1. Motivating Observations : Local Instability and Cycles

Introductory example

Figure 9: Estimated Nonlinear and Linear Model Impulse Response



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

- This suggests a specific way of looking at the data
- estimation of

$$x_t = A(L)x_{t-1} + F(x_{t-1}, x_{t-2}, \dots) + \epsilon_t$$

- x?
  - × We favor intensive variables (hours, etc...)
  - × We filter out low frequencies

- This suggests a specific way of looking at the data
- estimation of

$$x_t = A(L)x_{t-1} + F(x_{t-1}, x_{t-2}, \dots) + \epsilon_t$$

- x?
- $A(L)$?
  - × Influenced by our previous theory work.
  - × Allow $A(L)x_{t-1} = a_1 x_{t-1} + a_2 x_{t-2} + a_3 \underbrace{\sum_{i=0}^{N} (1-\delta)^i x_{t-1-i}}_{X_{t-1}}$

- This suggests a specific way of looking at the data
- estimation of

$$x_t = A(L)x_{t-1} + F(x_{t-1}, x_{t-2}, \dots) + \epsilon_t$$

- x?
- $A(L)$?
- F?
  - × $F(\cdot)$ is a third order polynomial of (x_{t-1}, x_{t-2}, X_t).
  - × "Minimal" model

$$x_t = \beta_0 + \beta_x\ x_{t-1} + \beta_{x'}\ x_{t-2} + \beta_X\ X_{t-1} + \beta_{x^3} x_{t-1}^3 + \epsilon_t.$$

1. Motivating Observations : Local Instability and Cycles

Strategy

- This suggests a specific way of looking at the data
- estimation of

$$x_t = A(L)x_{t-1} + F(x_{t-1}, x_{t-2}, \dots) + \epsilon_t$$

- x?
- $A(L)$?
- F?
  - × $F(\cdot)$ is a third order polynomial of (x_{t-1}, x_{t-2}, X_t).
  - × "Full" model

$$\begin{aligned}
x_t = {} & \beta_0 + \beta_x\ x_{t-1} + \beta_{x'}\ x_{t-2} + \beta_X\ X_{t-1} + \beta_{x^2}\ x_{t-1}^2 + \beta_{x'^2}\ x_{t-2}^2 + \beta_{X^2}\ X_{t-1}^2 \\
& + \beta_{xx'}\ x_{t-1}x_{t-2} + \beta_{xX}\ x_{t-1}X_{t-1} + \beta_{x'X}\ x_{t-2}X_{t-1} \\
& + \beta_{x^3}\ x_{t-1}^3 + \beta_{x'^3}\ x_{t-2}^3 + \beta_{X^3}\ X_{t-1}^3 \\
& + \beta_{x^2x'}\ x_{t-1}^2x_{t-2} + \beta_{x^2X}\ x_{t-1}^2X_{t-1} \\
& + \beta_{x'^2X}\ x_{t-2}^2X_{t-1} + \beta_{xx'^2}\ x_{t-1}x_{t-2}^2 \\
& + \beta_{xX^2}\ x_{t-1}X_{t-1}^2 + \beta_{x'X^2}\ x_{t-2}X_{t-1}^2 \\
& + \beta_{xx'X}\ x_{t-1}x_{t-2}X_{t-1} + \epsilon_t.
\end{aligned}$$

- h: Total Hours Worked per Capita, U.S.A., 1960-2015
- High-Pass Filtered, 80 quarters
- "Minimal" model

$$\begin{cases} h_t &= \alpha_0 + \alpha_1 h_{t-1} + \alpha_2 h_{t-2} + \alpha_3 H_{t-1} + \alpha_4 h_{t-1}^3 + \varepsilon_t \\ H_t &= \sum_{j=0}^{N} (1-\delta)^j h_{t-j} \end{cases}$$

Estimated Reduced Form

$$\begin{cases} h_t &= -0.00 + 1.42\ h_{t-1} - 0.48\ h_{t-2}, \\ h_t &= -0.01 + 1.31\ h_{t-1} - 0.34\ h_{t-2} - 0.25\ H_{t-1}, \\ h_t &= -0.02 + 1.39\ h_{t-1} - 0.34\ h_{t-2} - 0.27\ H_{t-1} - 0.01\ h_{t-1}^3. \end{cases}$$

- Non-linear term is significant
- Non-linear term enters with a *minus*
-

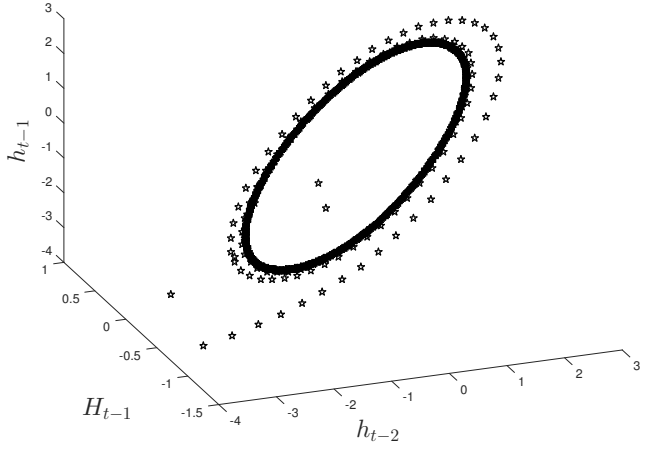
	$AR(2)$	Linear	Minimal
R^2	0.94	0.94	0.94
Max eig.	0.86	0.96	1.01

- R^2 is not much improved
- But max eigenvalue (in modulus) crosses 1 with the nonlinear term
- SS is unique, unstable

1. Motivating Observations : Local Instability and Cycles

Estimated Reduced Form - Total Hours

Figure 10: The Limit Cycle - Simulation as of $T_0 = 1961$



1. Motivating Observations : Local Instability and Cycles

Estimated Reduced Form - Total Hours

Figure 11: The Limit Cycle - Simulation as of $T_0 = 1961$

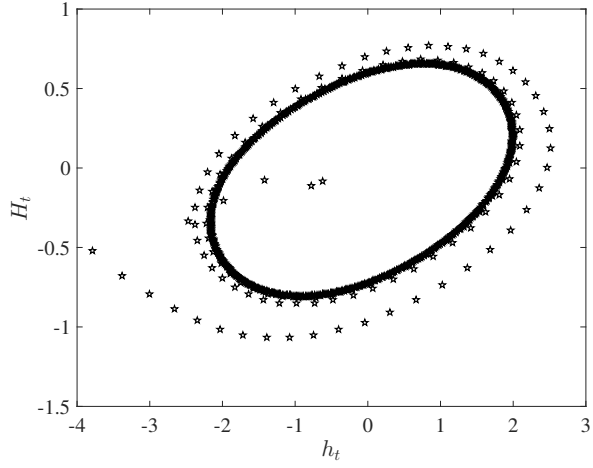
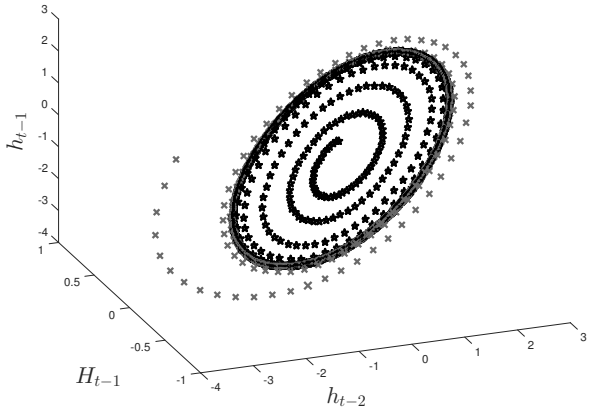


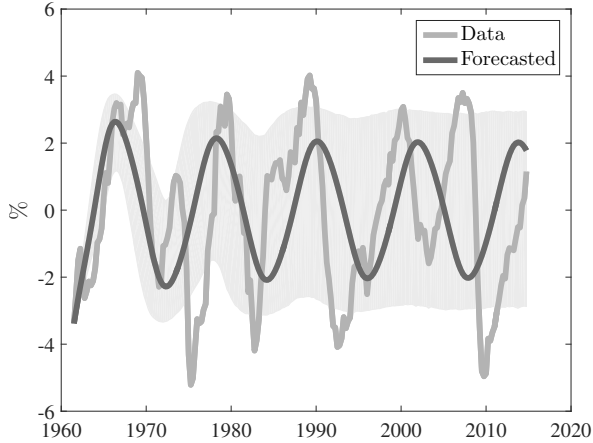
Figure 12: The Limit Cycle



1. Motivating Observations : Local Instability and Cycles

Estimated Reduced Form - Total Hours

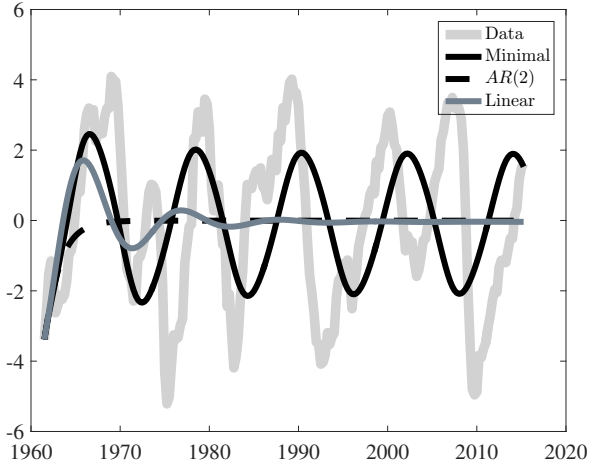
Figure 13: Forecasted Path as of 1961Q3 with the Minimal Model, Total Hours



1. Motivating Observations : Local Instability and Cycles

Estimated Reduced Form - Total Hours

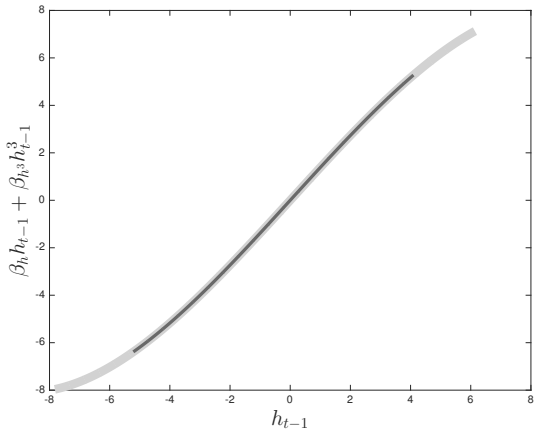
Figure 14: Forecasted Path as of 1961Q3, Total Hours



1. Motivating Observations : Local Instability and Cycles

Estimated Reduced Form - Total Hours

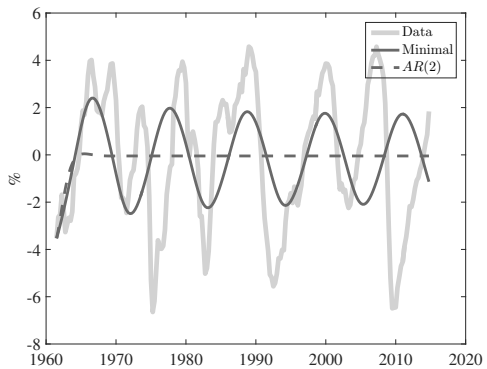
Figure 15: Nonlinearities in the Minimal Model, Total Hours



$$h_t = -0.02 + 1.39\ h_{t-1} - .34\ h_{t-2} - .27 H_t - .01\ h_{t-1}^3 + \epsilon_t$$

Figure 16: The Limit Cycle in the Minimal Model, Non-Farm Business Hours

(a) Against time



(b) Projection on (h_t, H_t)

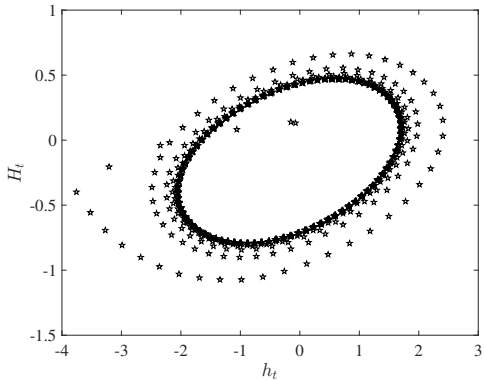
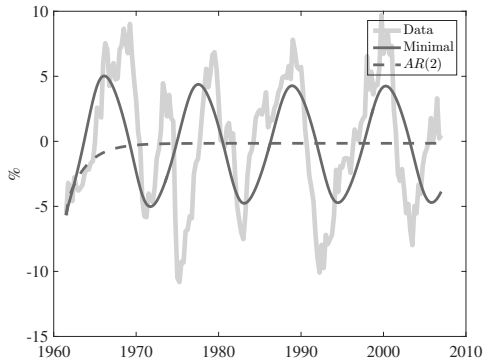


Figure 17: The Limit Cycle in the Minimal Model, Job Finding Rate

(c) Against time



(d) Projection on (p_t, P_t)

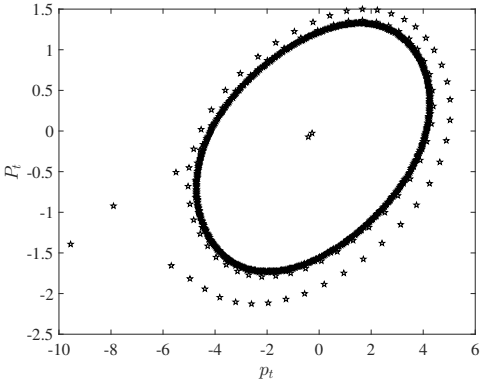
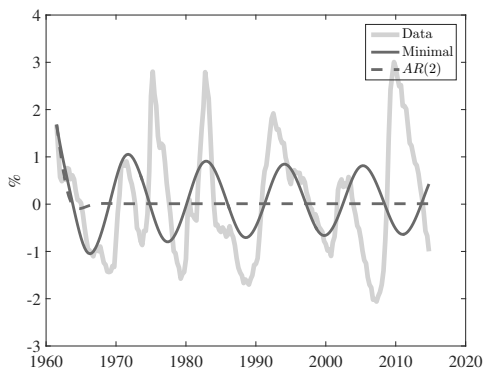


Figure 18: The Limit Cycle in the Minimal Model, Unemployment Rate

(e) Against time



(f) Projection on (u_t, U_t)

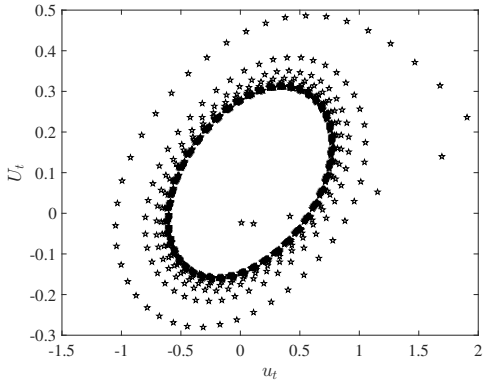


Figure 19: Unemployment Rate, The Netherlands

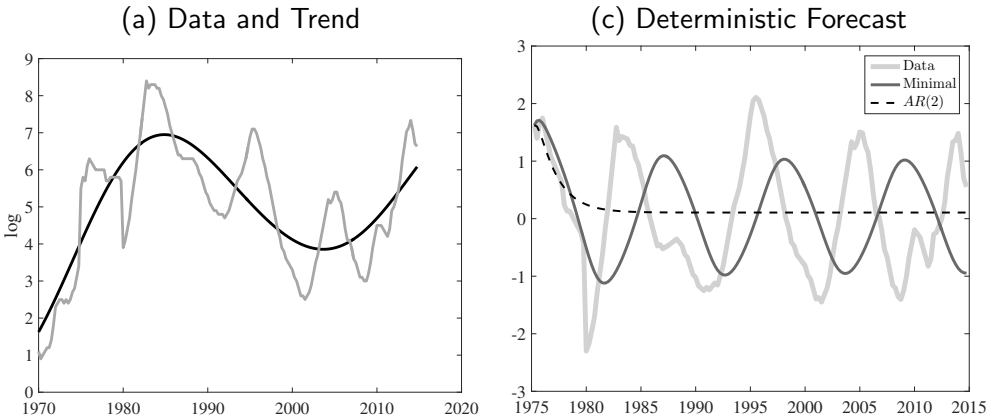
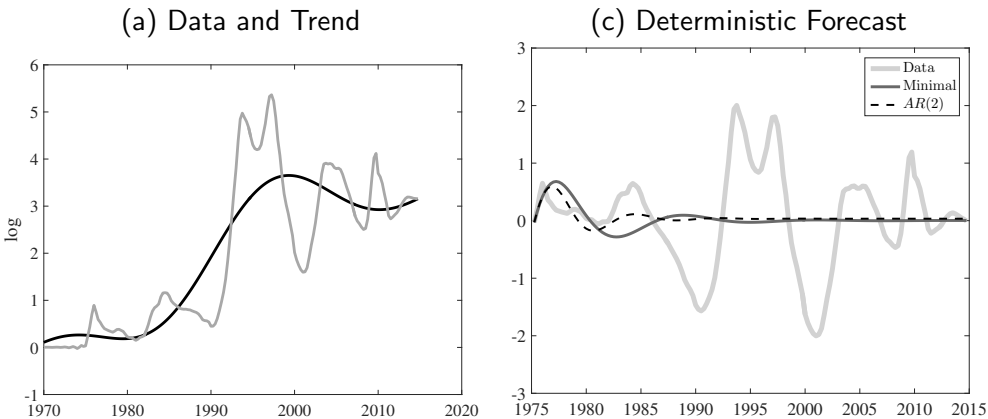


Figure 20: Unemployment Rate, Switzerland



Roadmap

1. Motivating Observations : Local Instability and Cycles
2. A Reduced Form (Agent Based) Approach
3. Structural Approach

Roadmap

1. Motivating Observations : Local Instability and Cycles
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3. Structural Approach

2. A Reduced Form (Agent Based) Approach

Environment

- N players
- Each agent accumulates X_i by playing e_i
- Decision rule and law of motion for X are

$$X_{it+1} = (1-\delta)X_{it} + e_{it} \tag{3}$$

$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t) \tag{4}$$

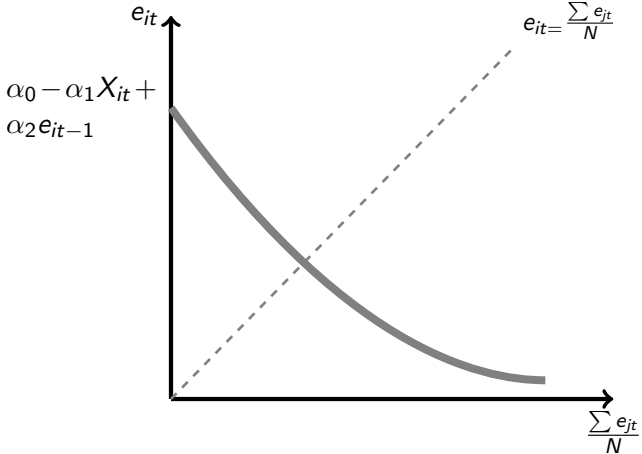
  with $e_t = \frac{\sum e_{jt}}{N}$
- all parameters are positive, $\delta < 1$, $\alpha_1 < 1$, $\alpha_2 < 1$.
- $\alpha_1 X_{it}$: Optimal size argument
- $\alpha_2 e_{it-1}$: Adjustment cost argument
- $F(e_t)$: Strategic Interactions
  - $F' > 0$: strategic complementarities
  - $F' < 0$: strategic substitutabilities

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t) \end{aligned}$$

- Define (X^s, e^s) as the steady state of the linear model where $F(\cdot) = 0$
- Normalize $F(e^s) = 0$

2. A Reduced Form (Agent Based) Approach

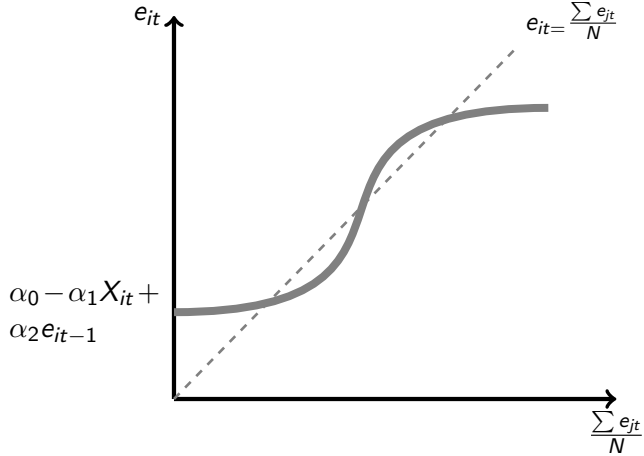
"Best response" rule for a given history - Strategic substitutability



$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

2. A Reduced Form (Agent Based) Approach

"Best response" rule for a given history - Multiple Equilibria

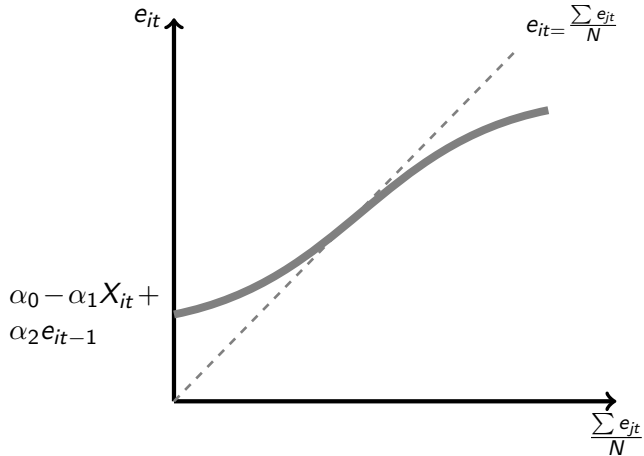


$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

2. A Reduced Form (Agent Based) Approach

"Best response" rule for a given history - Strategic complementarity

- We restrict to the case where $F'(\cdot) < 1$, so that there are never multiple equlibria *within* period t



- Assume $\alpha_2 = 0$ in

$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

- Then the model is

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 X_{it} + F(e_t) \end{aligned}$$

- It boils down to one order one dynamic equation in X_{it}

$$X_{it+1} = \alpha_0 + (1 - \delta - \alpha_1)X_{it} + F(X_{t+1} - (1-\delta)X_t)$$

- So that, for symmetrical allocations

$$X_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)X_t + F(X_{t+1} - (1-\delta)X_t)$$

- Assume no strategic interactions : $F(\cdot) \equiv 0$

$$X_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)X_t$$

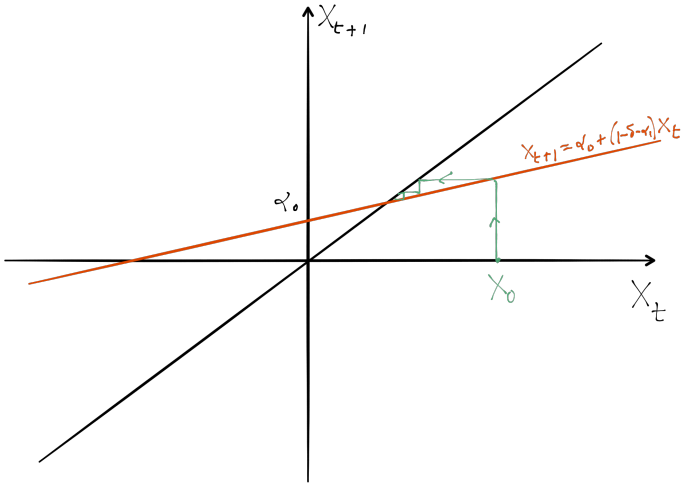
- Note that $(1 - \delta - \alpha_1)$ can be either > 0 or < 0.

Proposition 1

The Economy is globally stable as $|1 - \delta - \alpha_1| < 1$

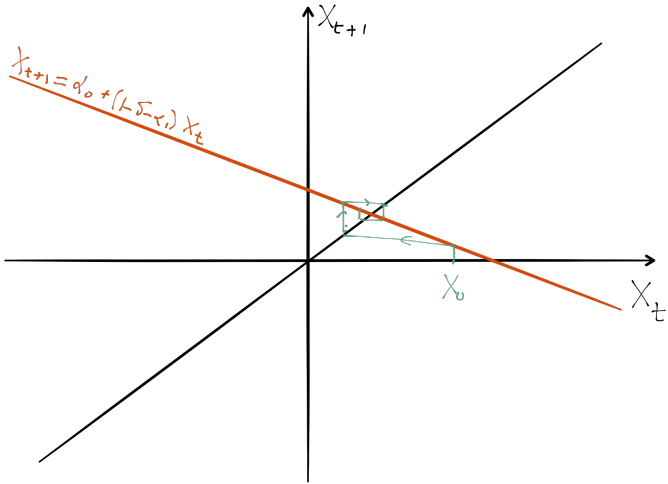
2. A Reduced Form (Agent Based) Approach

One-Dimension Case



2. A Reduced Form (Agent Based) Approach

One-Dimension Case



- Reintroduce strategic interactions

$$X_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)X_t + F(X_{t+1} - (1 - \delta)X_t)$$

- Linearize around the steady state X^s

$$X_{t+1} = (\alpha_0 + \delta X^s) + (1 - \delta - \alpha_1)X_t + F'(e^s)(X_{t+1} - (1 - \delta)X_t)$$

- so that

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(e^s)} + \frac{((1 - \delta)(1 - F'(e^s)) - \alpha_1)}{1 - F'(e^s)} \; X_t$$

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(e^s)} + \frac{((1-\delta)(1-F'(e^s)) - \alpha_1)}{1 - F'(e^s)} \quad X_t$$

Proposition 2

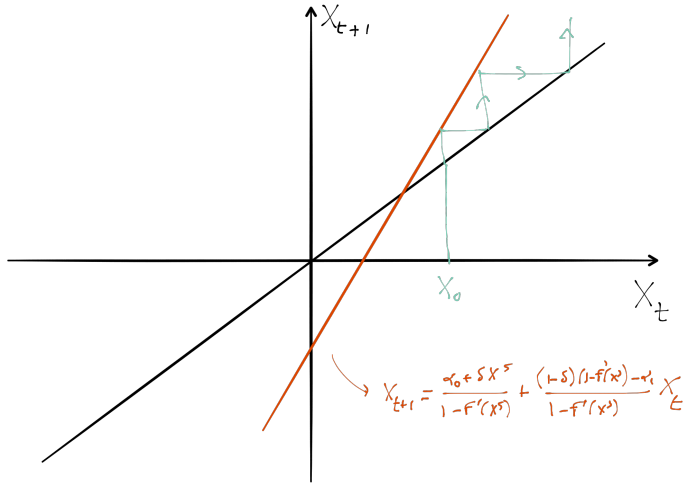
With strategic substitutability ($F'(\cdot) < 0$), the economy is always stable

Proposition 3

With strategic complementarity ($F'(\cdot) > 0$), there is always a level of complementarity smaller than 1 such that the SS is (locally) unstable.

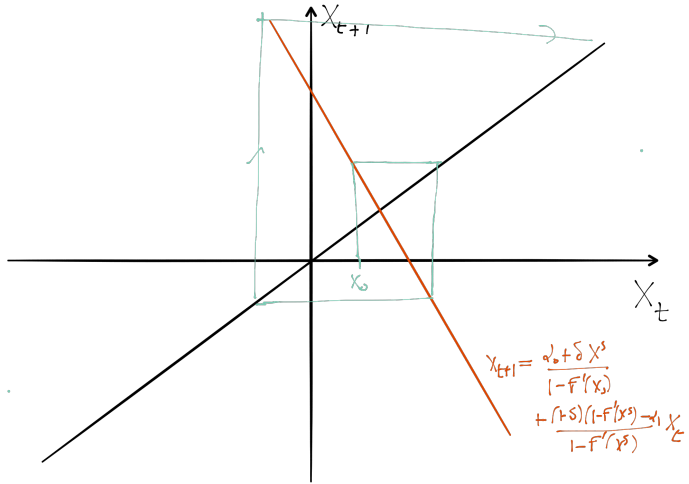
2. A Reduced Form (Agent Based) Approach

One-Dimension Case

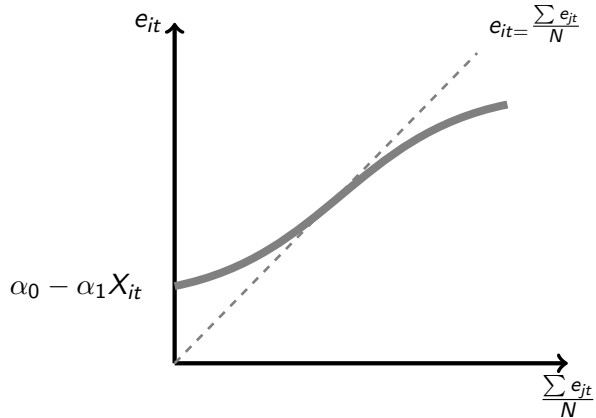


2. A Reduced Form (Agent Based) Approach

One-Dimension Case



- The SS X^s becomes locally unstable
- "Real data" do not look explosive
- Let's assume that strategic complementarities dies out when the economy is far above of below X^s
- F is "S-shaped"



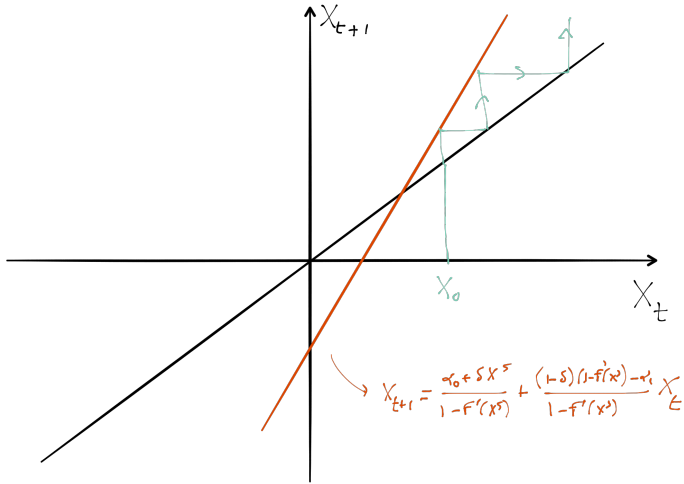
- Assume for simplicity that F is symmetric wrt X^s

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(e^s)} + \frac{((1-\delta)(1-F'(e^s)) - \alpha_1)}{1 - F'(e^s)} \; X_t$$

- The dynamics of the economy will very much depend on whether $\frac{((1-\delta)(1-F'(e^s))-\alpha_1)}{1-F'(e^s)}$ is positive or negative
  - × Positive: Hysteresis
  - × Negative : Limit cycle
- Need to think of environments that makes this slope is positive or negative

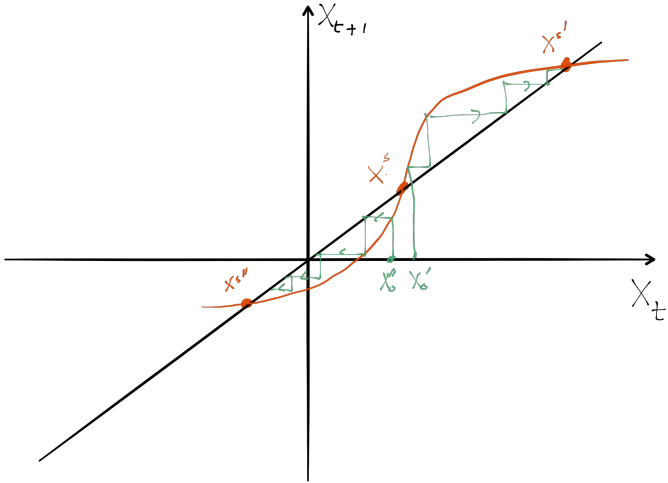
2. A Reduced Form (Agent Based) Approach

One-Dimension Case: Adding Nonlinearities



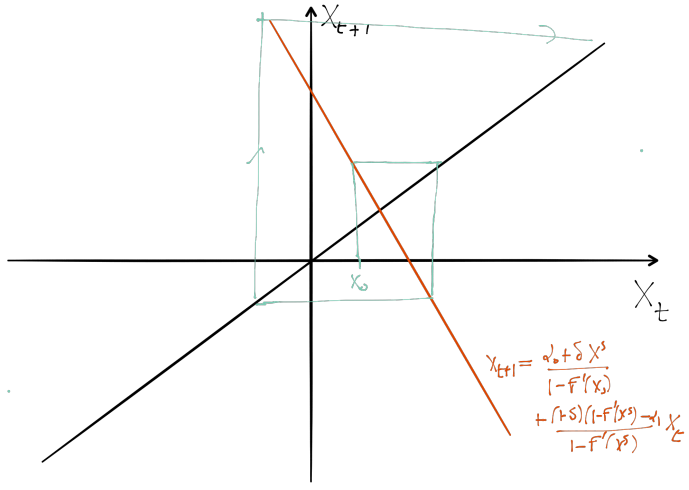
2. A Reduced Form (Agent Based) Approach

One-Dimension Case: Adding Nonlinearities



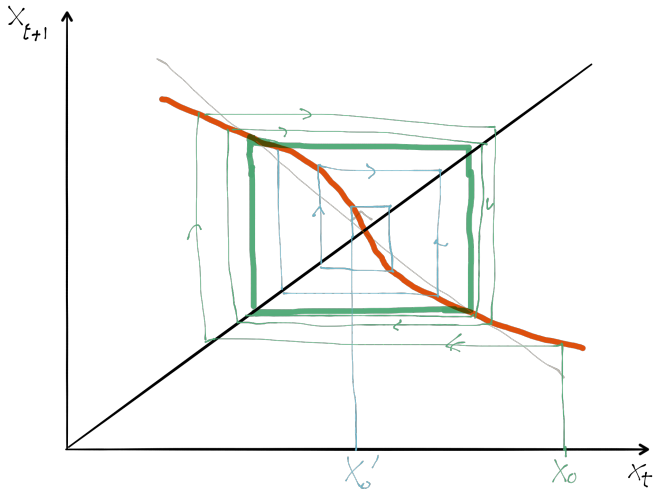
2. A Reduced Form (Agent Based) Approach

One-Dimension Case: Adding Nonlinearities



2. A Reduced Form (Agent Based) Approach

One-Dimension Case: Adding Nonlinearities



- Problem with one dimension model: the dynamics does not look like a business cycle
- X_t is *not* positively correlated.
- This needs not to be true in the more general model where $\alpha_2 > 0$

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t) \end{aligned}$$

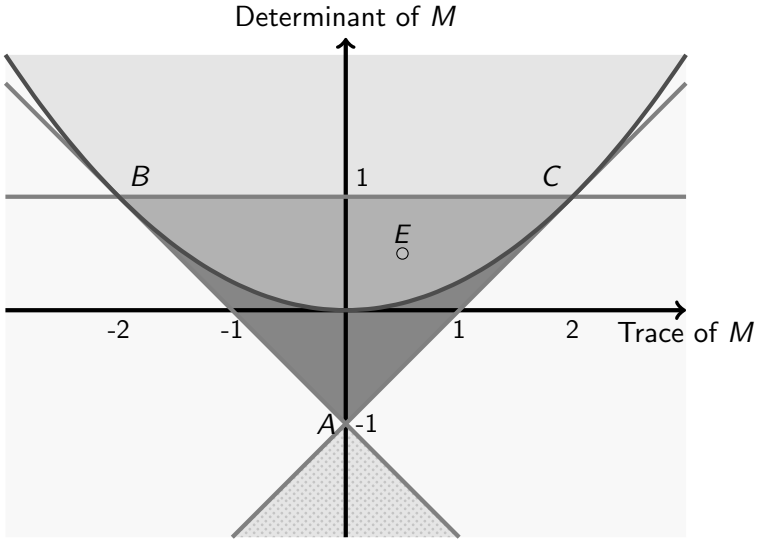
- Without strategic interactions ($F(\cdot) \equiv 0$)
- When all agent behave symmetrically:

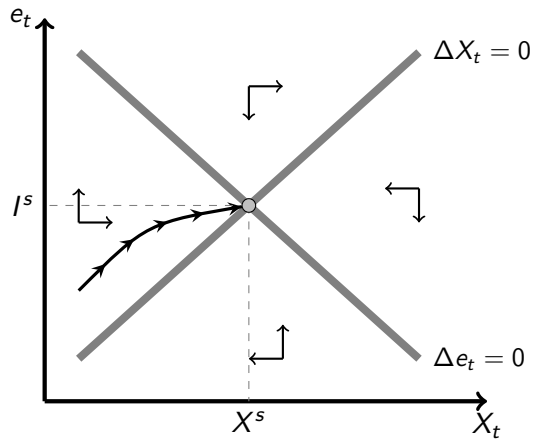
$$\begin{pmatrix} e_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_2 - \alpha_1 & -\alpha_1(1-\delta) \\ 1 & 1-\delta \end{pmatrix}}_{M} \begin{pmatrix} e_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}$$

Proposition 4

Both eigenvalues of the matrix M lie within the unit circle. Therefore, the system is stable.

Figure 21: Stability of the SS





Here we have assumed that the two eigenvalues are real and positive.

- Math
- Dynamics is

$$\begin{pmatrix} e_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_2 - \alpha_1 & -\alpha_1(1-\delta) \\ 1 & 1-\delta \end{pmatrix}}_{M_L} \begin{pmatrix} e_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_0 + F(e_t) \\ 0 \end{pmatrix}$$

- Local dynamics is

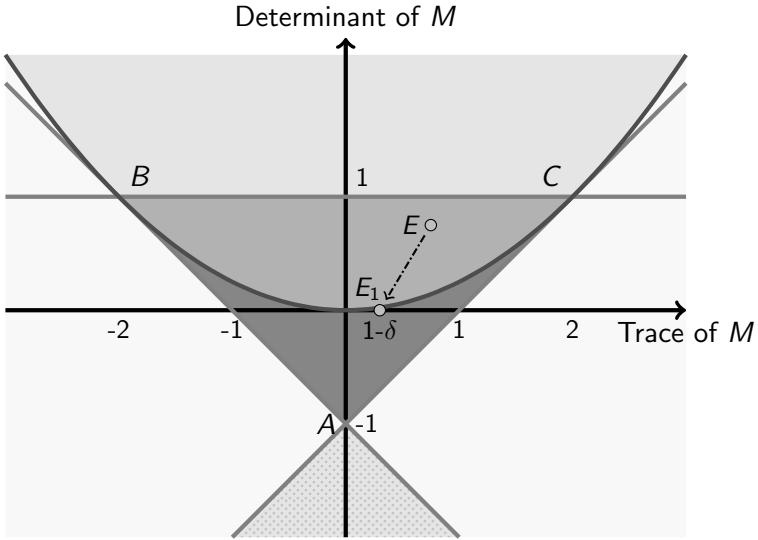
$$\begin{pmatrix} e_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\alpha_2-\alpha_1}{1-F'(e^s)} & -\frac{\alpha_1(1-\delta)}{1-F'(e^s)} \\ 1 & 1-\delta \end{pmatrix}}_{M} \begin{pmatrix} e_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \left(1 - \frac{\alpha_2-\alpha_1}{1-F'(e^s)}\right) e^s + \left(\frac{\alpha_1(1-\delta)}{1-F'(e^s)}\right) X^s \\ 0 \end{pmatrix}$$

- When $F'(e^s)$ varies from $-\infty$ to 1, eigenvalues of M vary

Proposition 5

As $F'(e^s)$ varies from 0 to $-\infty$, the eigenvalues of M always stay within the unit circle and therefore the system remains locally stable.

Figure 22: Strategic substitutabilities – F' negative



Proposition 6

As $F'(e^s)$ *varies from* 0 *towards* 1, *the dynamic system will become locally unstable.*

(bifurcation)

- 3 types of bifurcation
  - Fold bifurcation: appearance of an eigenvalue equal to 1,
  - Flip bifurcation: appearance of an eigenvalue equal to -1
  - HOPF bifurcation: appearance of two complex conjugate eigenvalues of modulus 1.

Figure 23: Strategic complemetarities – F' positive ($\alpha_2 > \alpha_1$)

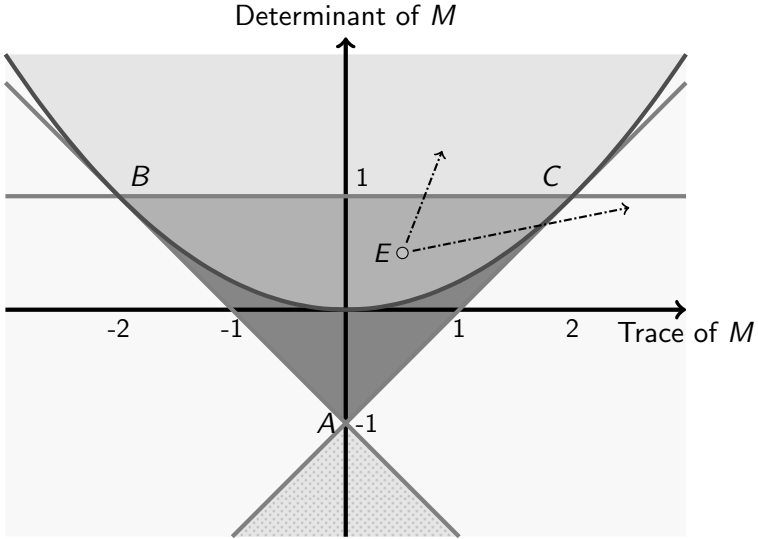
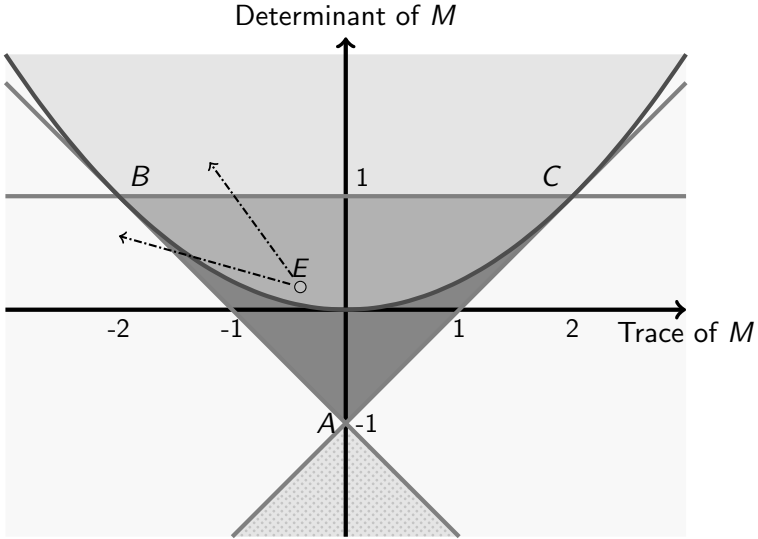


Figure 24: Strategic complemetarities – F' positive ($\alpha_2 < \alpha_1$)



- We are interested in HOPF bifurcation because the limit cycle will be "persistent"

Proposition 7

As $F'(e^s)$ varies from 0 towards 1, the dynamic system will become unstable and we'll have smooth cycles (HOPF (NEIMARK-SACKER)) if α_2 is large enough.

$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t)$$

- Discrete time version of the HOPF theorem.
- Nice theorem: we simply have to look at the *linearized* dynamics to prove existence of a limit cycle
- The parameter that varies is here the degree of strategic complementarities at the steady state $F'(e^s)$
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable

- Here we can have a limit cycle with persistence
- Consider the steady state (X^s, e^s)
- Strategic complementarities: centrifugal force that pushes away from the steady state when close to.
- Accumulated variable X: centripetal force that pushes towards the steady state when away from.
- The steady state locally unstable, but forces push the economy back to the steady state when it is further from it.
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable
- In the case of the HOPF bifurcation, the limit cycle can be attractive (*the bifurcation is supercritical*) or repulsive (*the bifurcation is subcritical*)

Figure 25: Stable Steady State

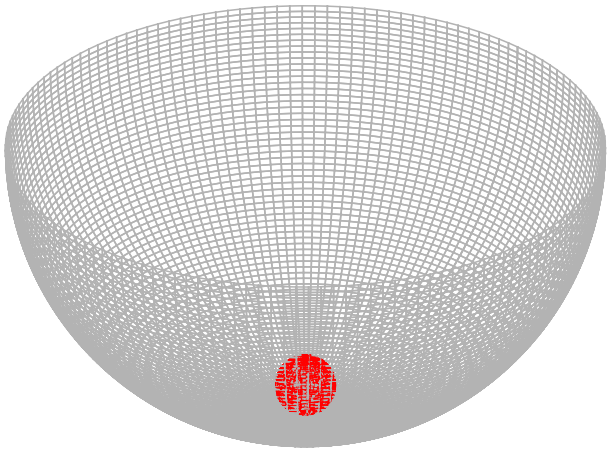


Figure 26: HOPF Supercritical bifurcation: Attractive Limit Cycle

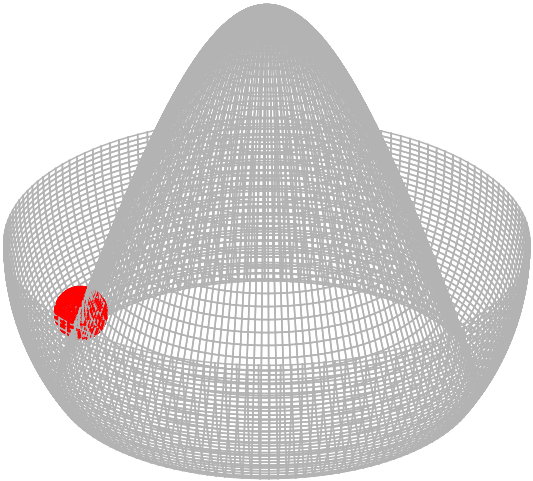


Figure 27: Unstable Steady State

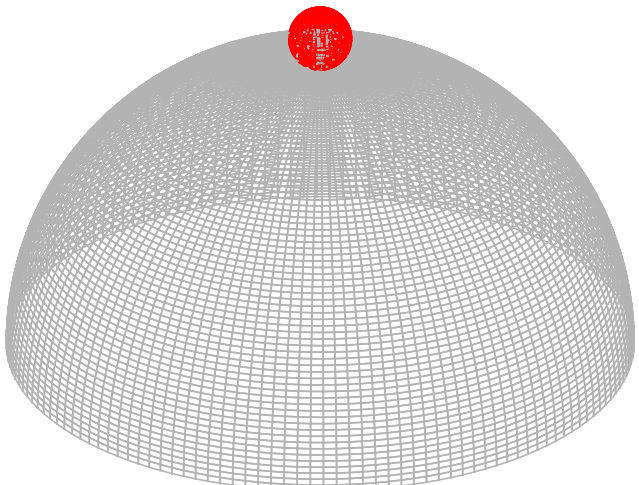
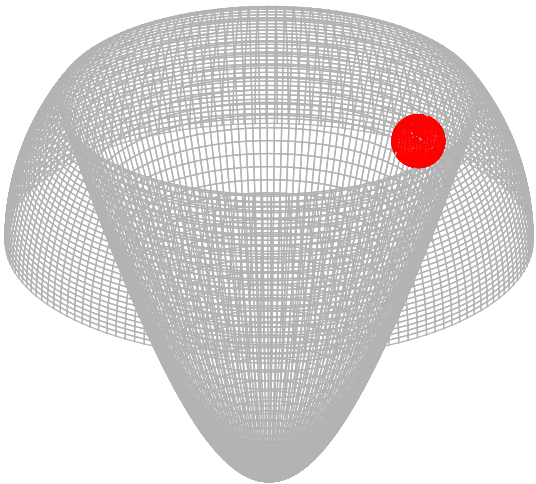
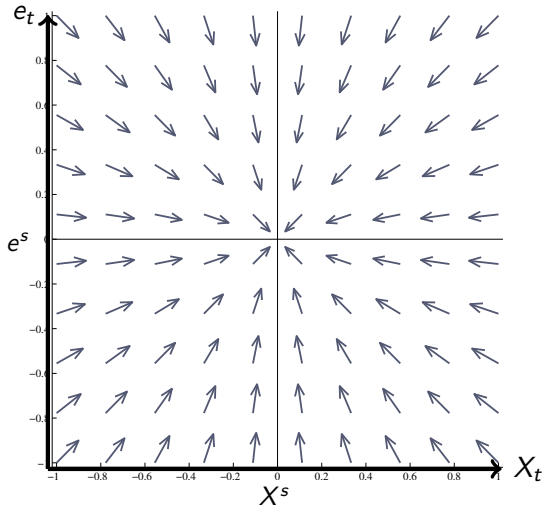


Figure 28: HOPF Subcritical bifurcation: Repulsive Limit Cycle



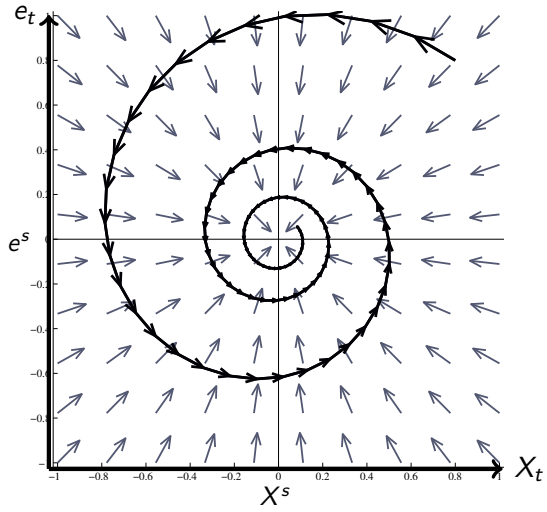
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Global stability



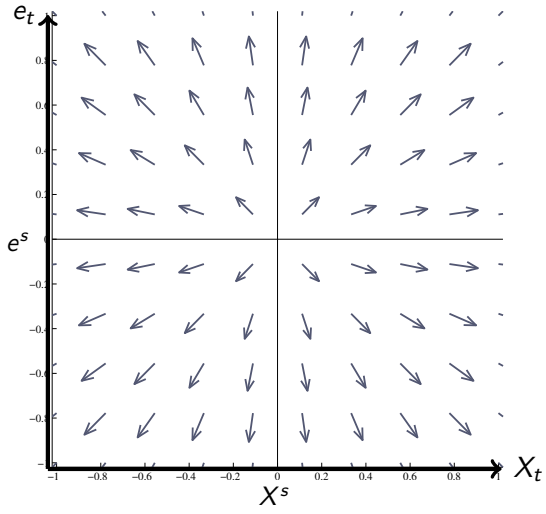
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Global stability



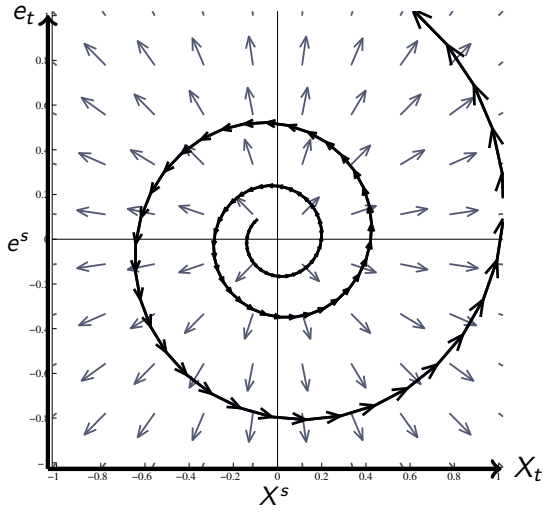
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Global instability



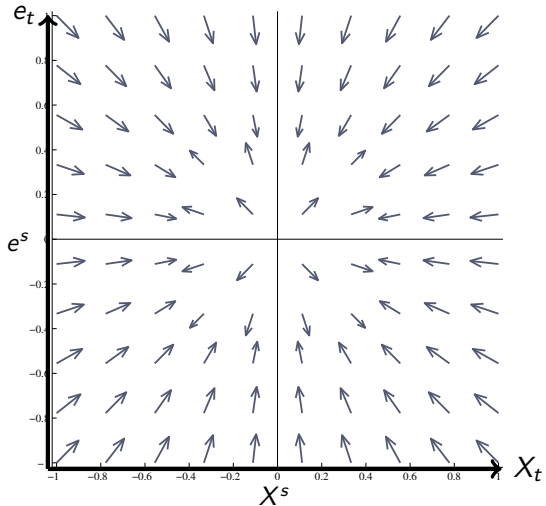
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Global instability



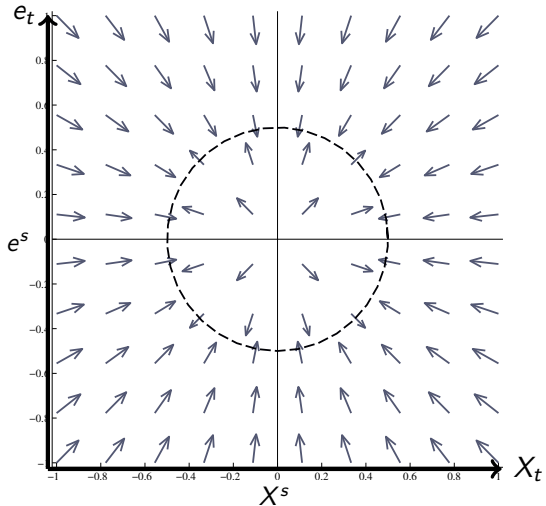
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Stable limit cycle



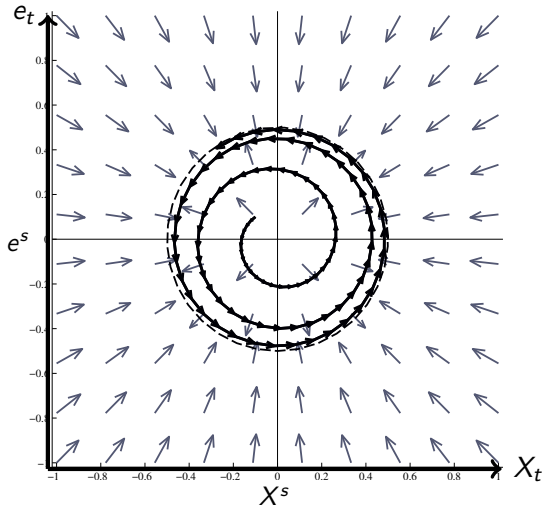
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Stable limit cycle



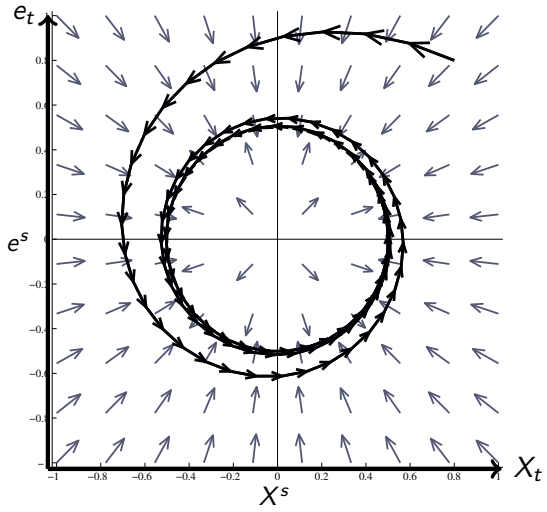
2. A Reduced Form (Agent Based) Approach

Two-Dimension Case - Stable limit cycle



2. A Reduced Form (Agent Based) Approach

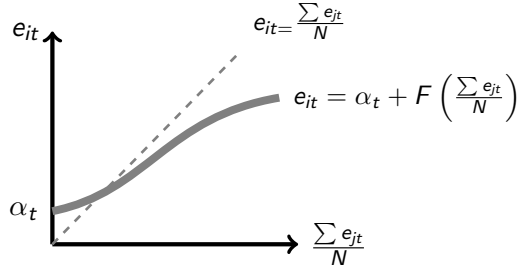
Two-Dimension Case - Stable limit cycle



Proposition 8

If $F'''(e^s)$ is sufficiency negative, then the HOPF *bifurcation will be supercritical. Therefore, the limit cycle is attractive.*

-
- $F''' < 0$ corresponds to an $S-$*shaped* reaction function

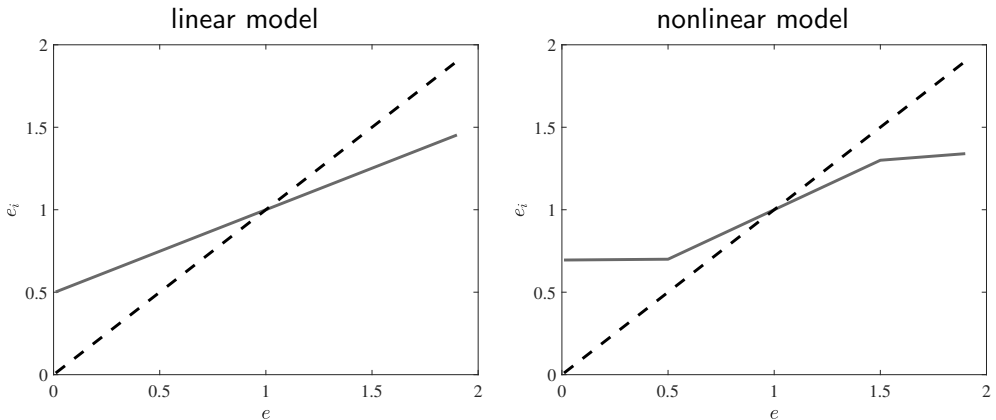


-

$$\alpha_t = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1}$$

- $X_{it+1} = (1-\delta)X_{it} + e_{it}$
- $e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{it}^s}{N}, u_t\right)$
- $F(e_t, u_t) = \widetilde{F}(e_t) + u_t.$
- $u_t = \rho u_{t-1} + \varepsilon_t$
- $\widetilde{F}$: S-shaped and piecewise linear

Figure 29: Best response rules in the numerical example



- Steady state is $e_{SS} = 1$, $X_{SS} = 10$
- Deterministic simulation: let $X_0 = 8$, $e_0 = 1$

Figure 30: Deterministic simulation

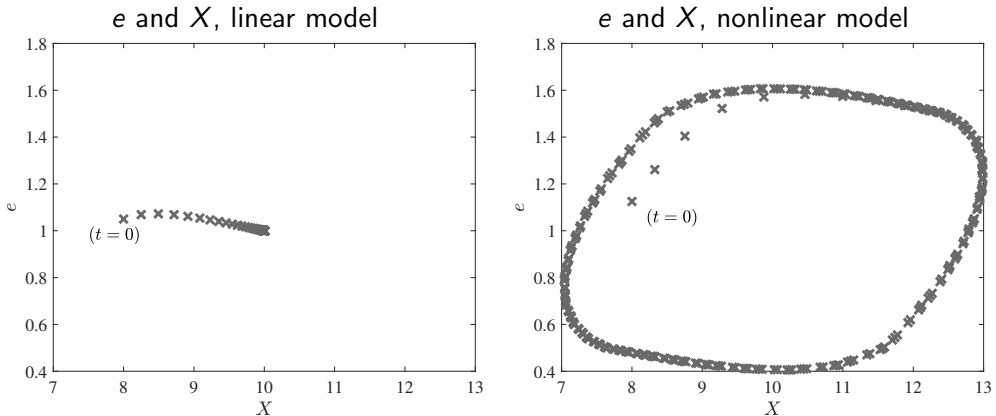


Figure 31: Deterministic simulation

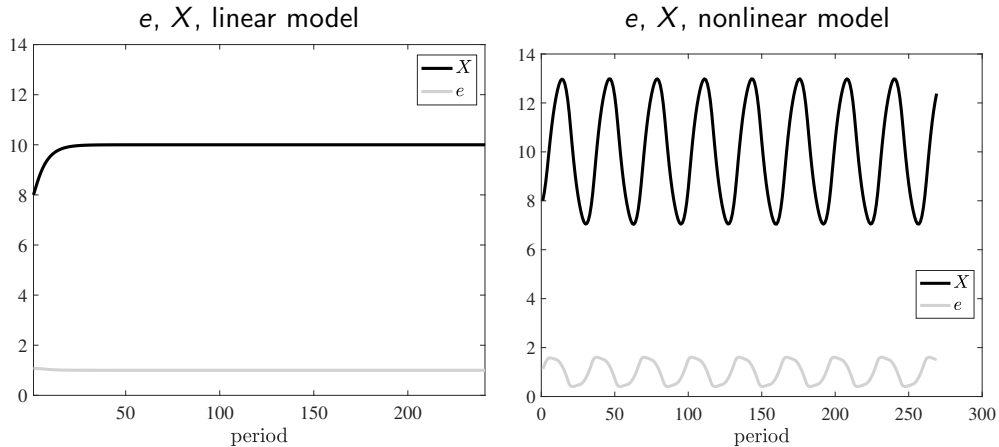


Figure 32: One stochastic simulation

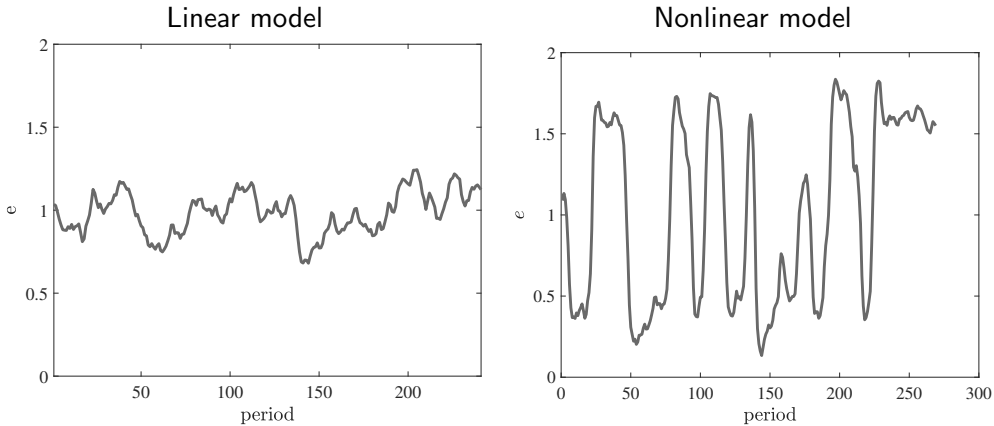


Figure 33: $e_t = \sin(\omega t)$

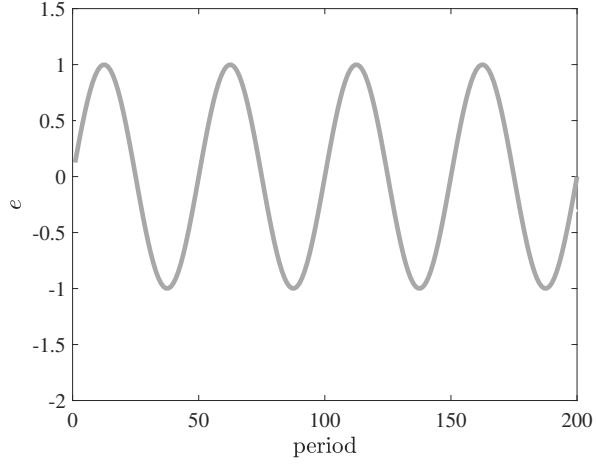


Figure 34: What the results are not: $e_t = \sin(\omega t) + u_t$

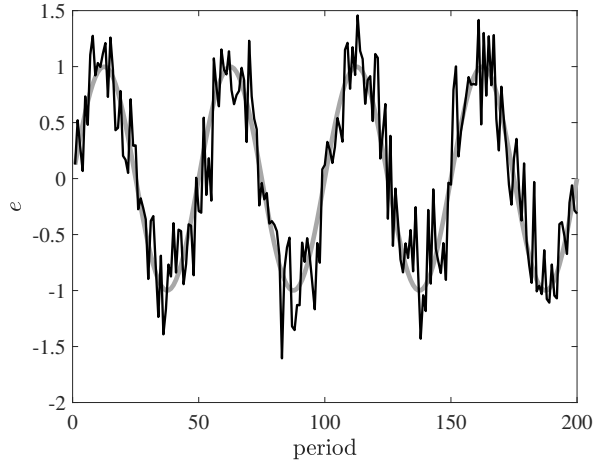


Figure 35: What the results are:

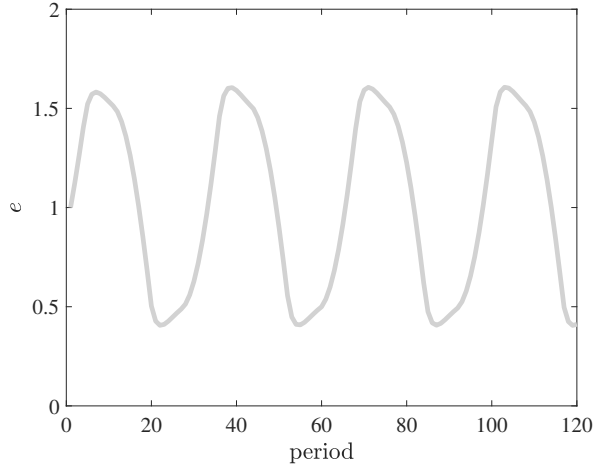
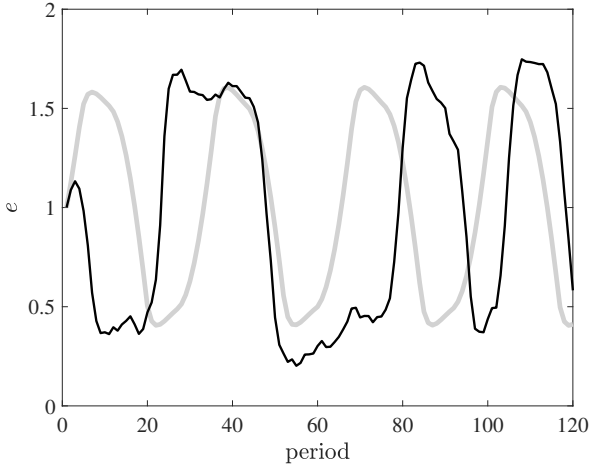


Figure 36: What the results are:



- $$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + \alpha_3 E_t[e_{it+1}] + F\left(\frac{\sum e_{jt}}{N}\right)$$
- with accumulation remaining the same

$$X_{it} = (1-\delta)X_{it} + e_{it}$$

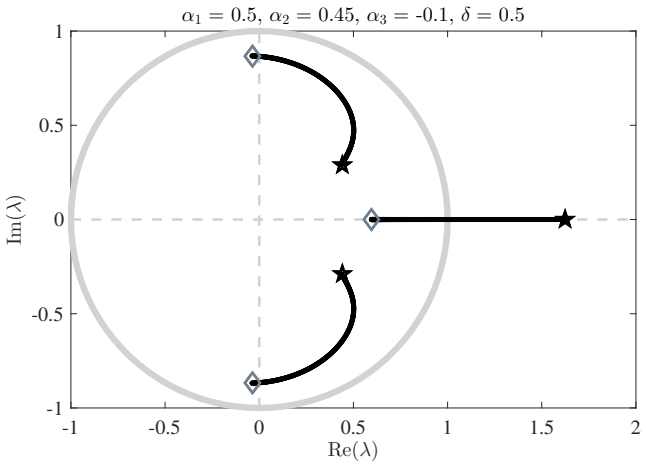
- Restrict attention to situations where this system is saddle path stable absent of complementarities.
- The local dynamics is described by the 3 eigenvalues of the linearized system

- Initial situation has two stable roots and one unstable
- Three types of bifurcations are possible:
  1. The unstable root enters the unit circle: local indeterminacy arises ("Benhabib-Farmer ")
  2. One stable root leaves the unit circle: instability arises with a flip or fold type bifurcation
  3. Two stable roots leave the unit circle simultaneous because they are complex: this is a Hopf bifurcation

2. A Reduced Form (Agent Based) Approach

Set of potential bifurcation with Forward looking elements

Figure 37: Eigenvalues of the Reduced Form Model

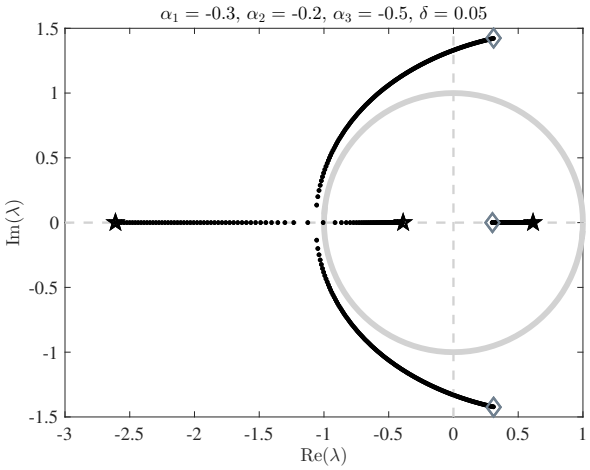


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2. A Reduced Form (Agent Based) Approach

Set of potential bifurcation with Forward looking elements

Figure 38: Eigenvalues of the Reduced Form Model

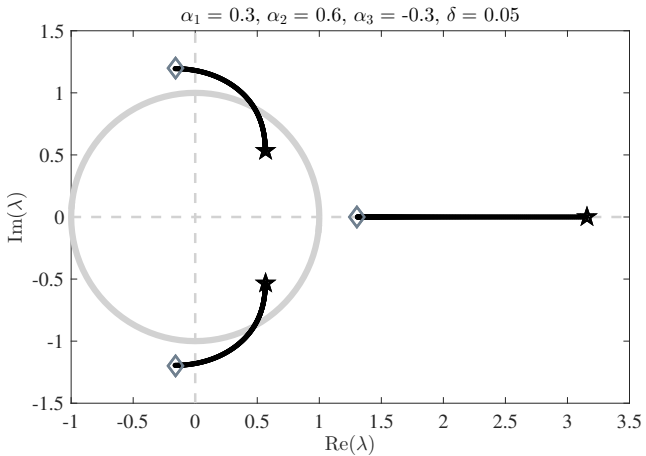


- Initial situation has two stable roots and one unstable
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  1. The unstable root enters the unit circle: local indeterminacy arises ("Benhabib-Farmer ")
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2. A Reduced Form (Agent Based) Approach

Set of potential bifurcation with Forward looking elements

Figure 39: Eigenvalues of the Reduced Form Model



Proposition 9

If unique steady state, then no indeterminacy nor Fold bifurcations (always one positive and greater than one eigenvalue)

Proposition 10

If α_2 (sluggishness) sufficiently large, then the two other eigenvalues will become complex and

- *either will stay inside the unit disk (non locally explosive cycles)*
- *or will exit the unit disk (HOPF bifurcation and limit cycles)*

Figure 40: Increasing complementarities ρ: local stability

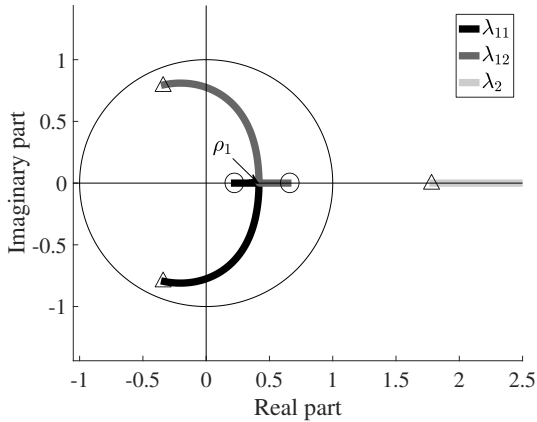


Figure 41: Increasing complementarities ρ: local instability and limit cycle

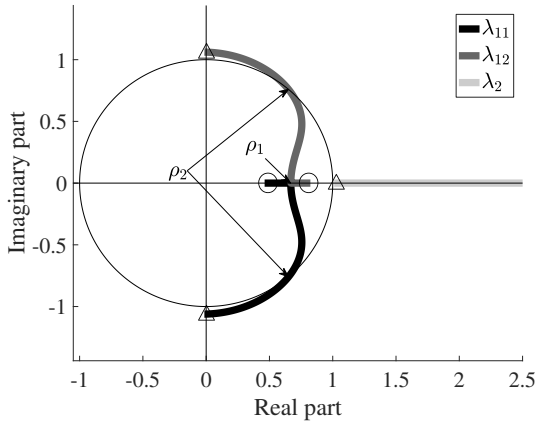


Figure 42: A Saddle Limit Cycle

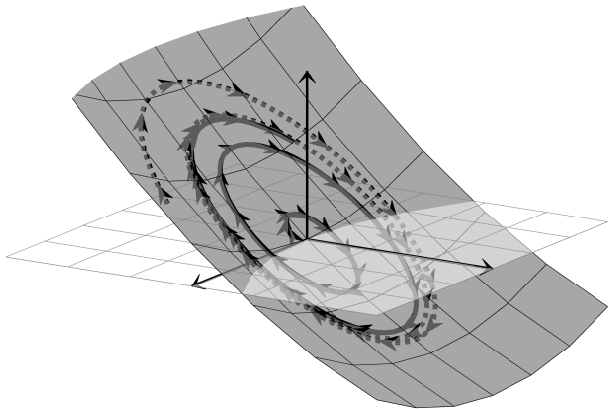


Figure 43: Explosive path

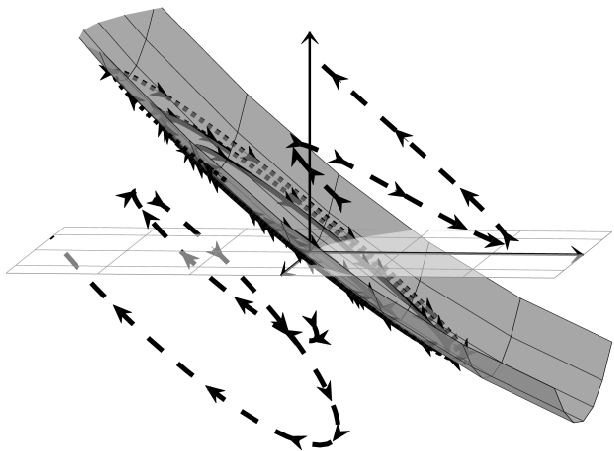
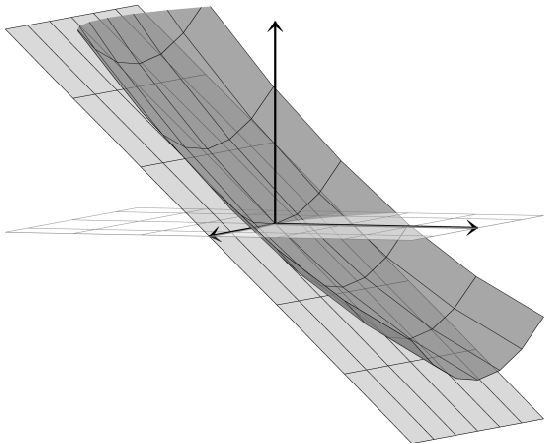


Figure 44: Linear and non linear stable manifolds



Roadmap

1. Motivating Observations : Local Instability and Cycles
2. A Reduced Form (Agent Based) Approach
3. Structural Approach

Roadmap

1. Motivating Observations : Local Instability and Cycles
2. A Reduced Form (Agent Based) Approach
3. Structural Approach

- Try to understand what type of agents interactions tend to cause what phenomena,
- Approach this with a few constraints
  1. Start with an agent based structure where, in the absence of interaction, outcomes are stable (unique steady state wit monotonic convergence)
  2. Want to understand emergence of fluctuations (or sensitivities) while simultaneously understanding why the economy appears globally quite stable
  3. Start by focusing on a maximizing agent setting with rational expectations
  4. Derive conditions of fundamentals (payoff functions)
- Approach: adopt a game theoretical structure that build on idea in Cooper and John (1988).

- The problem

$$\max \sum_{t=0}^{\infty} \beta^t E_t V(e_{it}, X_{it}; \bar{e}_t, \bar{X}_t, \theta_t)$$

- s.t.

$$X_{it+1} = \Gamma(X_{it-1}, e_{it}; \bar{e}_t, \bar{X}_t,)$$

- Question: how does the dependence of agent level payoff on the decisions of others affects the nature of dynamics.
- With the baseline being that if $\bar{e}$ and $\bar{X}$ where fixed, we would have monotone convergence to unique steady state.
- When $\beta = 0$, easy to get our "*Agent Based*" setup.

- In many cases, with an appropriate change in variables, this framework can be simplified to

$$\max \sum_{t=0}^{\infty} \beta^t E_t V(e_{it}, X_{it}; \bar{e}_t, \bar{X}_t, \theta_t)$$

- s.t.

$$X_{it+1} = (1-\delta)X_{it-1} + \psi e_{it}$$

- The agents behaviour can then be summarized by an Euler Equation of the form

$$V_1(e_{it}, X_{it}; \bar{e}_t, \bar{X}_t) = E_t \beta \Big[(1-\delta) V_1(e_{it+1}, X_{it+1}; \bar{e}_{t+1}, \bar{X}_{t+1}) - V_2(e_{it+1}, X_{it+1}; \bar{e}_{t+1}, \bar{X}_{t+1}) \Big]$$

- We are working on general propositions
- Today I will show an fully specified micro-founded General Equilibrium example
- The example can be thought as a variation of Benhabib & Farmer.

- Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \left(\log \left[\psi d_t + c_t - (1-\psi)\gamma h_t^{\mu} \overline{h}_t^{1-\mu} + \widetilde{s} \right] - \frac{\omega_0}{1+\omega} \ell^{1+\omega} \right),$$

- Technology is

$$d_{t+1} = (1-\delta_d) d_t + \gamma_d c_t,$$
$$h_{t+1} = (1-\delta_h) ht + \gamma_h c_t,$$

and

$$c_t = \chi_t \ell_t^{\alpha}.$$

- d_t represent a stock of durables and h_t stands for a stock of habit.
- Externalities in production are modelled as

$$\chi_t = \overline{c}_t = \chi \overline{\ell}_t^{\nu\alpha}.$$

Table 1: Common Parameters Values

β	α	χ	ω	γ_d	γ_h	$\widetilde{s}$
.98	.7	1	.5	.5	.5	1

Table 2: Specific Parameters Values

	ψ	γ	μ	δ_h	δ_d
Hopf case	.5	1	1	.9	.025
Fold case	.3	1	1	.05	.025
Sunspots case	.1	1	.3	.025	.025

Figure 45: HOPF Bifurcation When ν is Increased

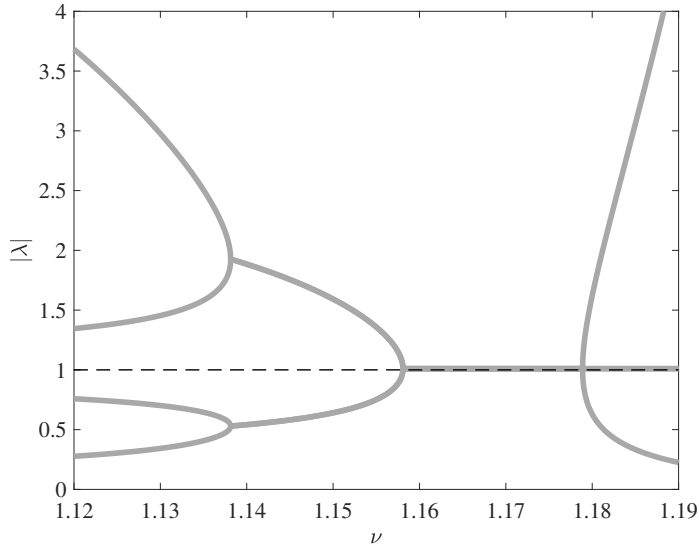


Figure 46: Fold Case Eigenvalues When ν is Increased

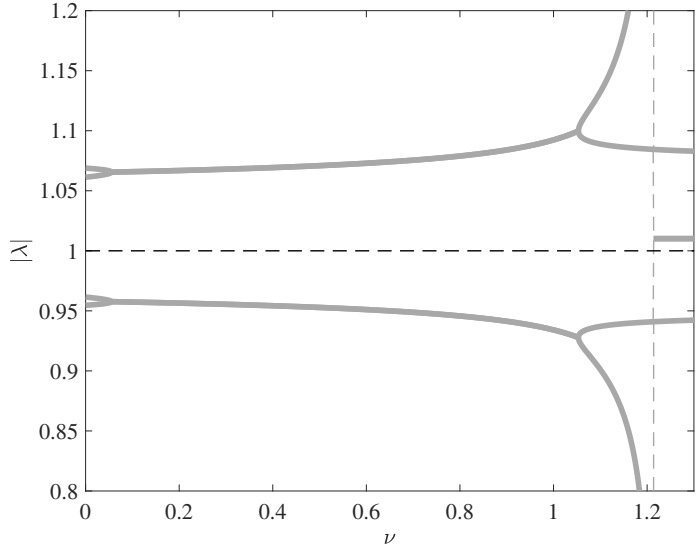


Figure 47: Steady State(s) in the Fold case

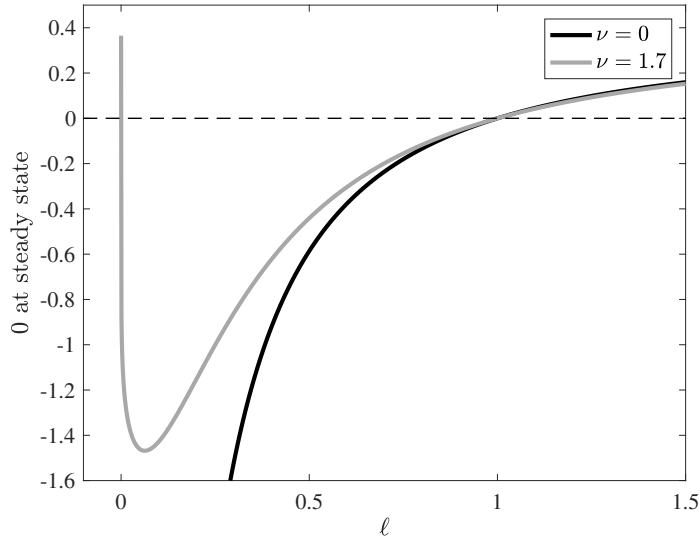
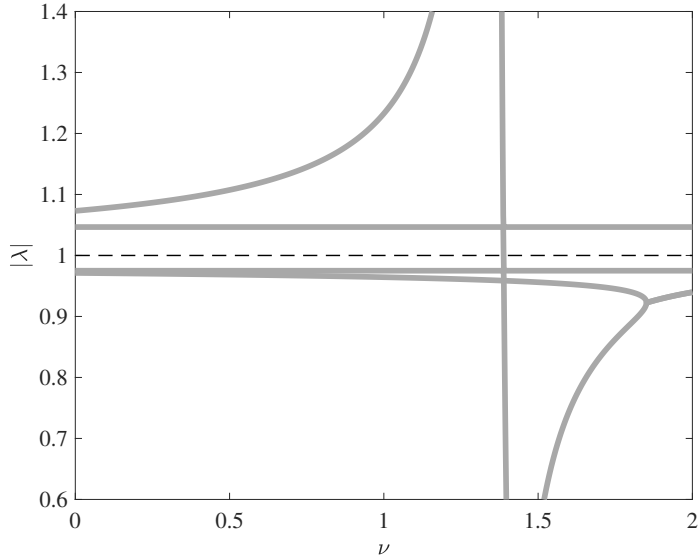


Figure 48: Bifurcation with Sunspots When ν is Increased



- More next time...

