University College London, 2018-2019
Econ 0107 – Macroeconomics I – Ralph Luetticke & Franck Portier

Midterm Exam

This is a 2 hours exam. No documents allowed. 100 points in total, 50 points for each problem.

I – General Equilibrium with Incomplete Markets

Consider the following setup: There is a continuum of households that have a felicity function over consumption, c, such that

$$u(c) = -\lambda^{-1}\exp(-\lambda c),$$

with $\lambda > 0$. Households face uninsurable income risk where income is given by

$$y \overset{i.i.d.}{\sim} N(\mu, \sigma).$$

Households have a time-preference rate of $\delta > 0$ and try to maximize

$$U = E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\delta}\right)^t u(c_t).$$

Households can self-insure (trading uncontingent claims amongst each other) against income risk using a risk-less asset a that bears interest r. There is no borrowing constraint except for the long-run transversality condition $\lim_{t\to\infty}(1+r)^{-t}a_t = 0$.

1 – Write the household's planning problem in form of a Bellman equation. Argue briefly, why the i.i.d. assumption for y allows the state space to be reduced to a single dimension.

The household budget constraint is given by

$$a' = (1+r)(a+y-c)$$

Define total ressources $x := a + y$, then we can rewrite the budget constraint as:

$$x' - y' = (x-c)(1+r).$$

We can use this to write the planning problem as

$$U(x) = \max_{a'} u[x - a'/(1+r)] + \frac{1}{1+\delta} EU(a'+y'). \tag{1}$$

The current income is no state variable, because of the independence of y' also $EU(a'+y')$ is independent of y conditional on knowing a'.

2 – Derive the Consumption Euler Equation and show that under the optimal policy the following must hold:

$$U = -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda[c_0]\}.$$

The Consumption Euler Equation is given by

$$u'(c_t) = \frac{1+r}{1+\delta} E_t u'(c_{t+1})$$

(Any standard way you like) as for CARA utility $u' = -\lambda u$, we have

$$u(c_t) = \frac{1+r}{1+\delta} E_t u(c_{t+1}).$$

Plugging this recursively in the definition of U, we obtain

$$U = E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\delta}\right)^t u(c_t) \tag{2}$$

$$= \sum_{t=0}^{\infty} \left(\frac{1}{1+\delta}\right)^t \left(\frac{1+\delta}{1+r}\right)^t u(c_0) = -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda c_0\} \tag{3}$$

3 – Guess and verify that the policy function $a(x)$, where $x = a + y$, takes the form

$$a'(x) = x - rA_0$$

Hints: (*i*) Show that under the guess x follows a Gaussian random walk (with drift), (*ii*) then, make use of the FOC derived in question **2**, (*iii*) use the result $E\exp(y) = \exp(\mu + \sigma^2/2)$.

Under this guess, we obtain

$$c(x) = x - a'(x)/(1+r) = \frac{r}{1+r}(x + A_0).$$

Moreover, x follows a random walk (with drift) as

$$x' = a'(x) + y' = x + y' - rA_0.$$

Note that this implies

$$U = -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda c(x)\} = -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\frac{r}{1+r}(x + A_0)\} \tag{4}$$

The above expression will be useful for question (7). This first-order condition writes:

$$-(1+r)^{-1}\exp\{-\lambda[x - a'/(1+r)]\} + \frac{1}{1+\delta}E\exp\{-\lambda\frac{r}{1+r}(a' + y' + A_0)\} = 0 \tag{5}$$

$$\exp\{-\lambda[x - a'/(1+r)]\} = \frac{1+r}{1+\delta}E\exp\{-\lambda\frac{r}{1+r}(a' + y' + A_0)\} \tag{6}$$

$$\exp\{-\lambda[x - a'/(1+r)]\} = \frac{1+r}{1+\delta}\exp\left(-\lambda\frac{r}{1+r}a'\right)\exp\left(-\lambda\frac{r}{1+r}A_0\right) \tag{7}$$

$$\exp\left[-\lambda\frac{r}{1+r}\mu + \left(\lambda\frac{r}{1+r}\sigma\right)^2/2\right]$$

$$-\lambda[x - a'/(1+r)] = \log\left(\frac{1+r}{1+\delta}\right) + \left(-\lambda\frac{r}{1+r}a'\right) + \left(-\lambda\frac{r}{1+r}A_0\right) \tag{8}$$

$$+\left[-\lambda\frac{r}{1+r}\mu + \left(\lambda\frac{r}{1+r}\sigma\right)^2/2\right]$$

$$a' = x + \lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) - \left(\frac{r}{1+r}A_0\right) - \left[\frac{r}{1+r}\mu - \lambda\left(\frac{r}{1+r}\sigma\right)^2/2\right] \tag{9}$$

This last expression is compatible with the guess.

4 – Solve for A_0. Show that for $r = \delta$ and $\lambda \neq 0$ assets drift to infinity. Interpret the expression for A_0 economically.

We need to solve

$$-rA_0 = \lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) - \left(\frac{r}{1+r}A_0\right) - \left[\frac{r}{1+r}\mu - \lambda\left(\frac{r}{1+r}\right)^2\right]$$

$$-(1+r)A_0 = \frac{1+r}{r}\lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) - A_0 - \left[\mu - \lambda\frac{r}{1+r}\sigma^2/2\right]$$

$$A_0 = -\frac{1+r}{r^2}\lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) + \frac{1}{r}\mu - \lambda\frac{1}{1+r}\sigma^2/2$$

For $\delta = r$ the first term vanishes. This yields for assets a

$$a' = a + y - \mu + \lambda\frac{r}{1+r}\sigma^2/2$$

Taking expectations we have $Ey = \mu$ and hence $Ea' = a + \lambda \frac{r}{1+r}\sigma^2/2 > a$. The higher the income risk σ or risk aversion, the higher expected savings $Ea' - a = \lambda \frac{r}{1+r}\sigma^2/2$.

5 – Define a "stationary equilibrium" for the above economy. Does it exist? If not how would you define a recursive equilibrium (with constant interest rate) here?

A stationary equilibrium is a distribution of assets and income $\phi(x)$, a value function $U(x)$, a policy function $a'(x)$ and an interest rate, such that (1) The distribution of assets and incomes remains constant over time, (2) at price r, $a'(x)$ is the optimal policy given $U(x)$, (3) $U(x)$ solves the corresponding Bellman equation and (4) the asset market clears: $\int x d\phi(x) = 0$.
Since assets follow a Gaussian random walk, there is no stationary distribution. I would define a recursive equilibrium (with constant r) here by replacing in the equilibrium objects "distribution ϕ" by a sequence of distribution and in the requirement (1) that this sequence is generated by the policy functions.

6 – Show that expression for the equilibrium interest rate(s) is approximately:

$$\left(\frac{1+r}{r}\right)^2 (\delta - r) = (\lambda\sigma)^2/2 \tag{10}$$

The equilibrium interest rate must lead to zero expected savings, i.e.

$$\begin{aligned} 0 = Ea' - a &= \mu - rA_0 && (11)\\ &= \mu - r\{-\frac{1+r}{r^2}\lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) + \frac{1}{r}\mu - \lambda\frac{1}{1+r}\sigma^2/2\} && (12)\\ &= -r\{-\frac{1+r}{r^2}\lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) - \lambda\frac{1}{1+r}\sigma^2/2\} && (13)\end{aligned}$$

Which implies

$$-\log\left(\frac{1+r}{1+\delta}\right) = \left(\lambda\frac{r}{1+r}\sigma\right)^2/2 \tag{14}$$

This is approximately

$$\left(\frac{1+r}{r}\right)^2 (\delta - r) = (\lambda\sigma)^2/2 \tag{15}$$

7 – For below questions, intuitive explanation would suffice. How many equilibria are there? Can you welfare rank them in general? If not, why? In that case, can you welfare rank them "behind a veil of ignorance" when nobody has accumulated claims against somebody else? How would you do this (w/o actual calculations!)?

Thus there are typically three equilibria. One cannot rank them in general, because the expected utility of a household is given by

$$-\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\frac{r}{1+r}(x + A_0(r))\}$$

and how this changes in r will depend on the asset position. An asset rich household prefers the high interest rate equilibrium. Behind a "veil of ignorance" and at $x = y$, we need to evaluate how (non-technical argument suffices)

$$\begin{aligned} & E - \lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\frac{r}{1+r}(y + A_0(r))\}\\ =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\frac{r}{1+r}(\mu - \lambda\frac{r}{1+r}\sigma^2/2 + A_0(r))\}\\ =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\frac{r}{1+r}(\mu - \lambda\frac{r}{1+r}\sigma^2/2 - \frac{1+r}{r^2}\lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right) + \frac{1}{r}\mu - \lambda\frac{1}{1+r}\sigma^2/2)\}\\ =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\frac{r}{1+r}(\frac{1+r}{r}\mu - \lambda\sigma^2/2 - \frac{1+r}{r^2}\lambda^{-1}\log\left(\frac{1+r}{1+\delta}\right)\}\\ =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\mu + \frac{r}{1+r}(\lambda\sigma)^2/2 + \frac{1}{r}\log\left(\frac{1+r}{1+\delta}\right)\}\end{aligned}$$

changes.
Plugging in the equilibrium condition

$$-\log\left(\frac{1+r}{1+\delta}\right) = \left(\lambda\frac{r}{1+r}\sigma\right)^2/2 \tag{16}$$

we obtain

$$\begin{aligned} EU =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\mu+\frac{r}{1+r}(\lambda\sigma)^2/2+\frac{1}{r}\log\left(\frac{1+r}{1+\delta}\right)\} \\ =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\mu+\frac{r}{1+r}(\lambda\sigma)^2/2-\frac{1}{r}\left(\lambda\frac{r}{1+r}\sigma\right)^2/2\} \\ =& -\lambda^{-1}\frac{1+r}{r}\exp\{-\lambda\mu+\frac{r^2}{(1+r)^2}(\lambda\sigma)^2/2\} \end{aligned}$$

Differentiating the expression w.r.t. r, we obtain

$$\frac{\partial EU}{\partial r} = \frac{1}{r^2}\lambda^{-1}\exp\{-\lambda\mu+\frac{r^2}{(1+r)^2}(\lambda\sigma)^2/2\} \tag{17}$$

$$-\frac{1}{(1+r)^2}(\lambda\sigma)^2\lambda^{-1}\exp\{-\lambda\mu+\frac{r^2}{(1+r)^2}(\lambda\sigma)^2/2\} \tag{18}$$

$$= EU\frac{r}{1+r}\left(\frac{1}{(1+r)^2}(\lambda\sigma)^2-\frac{1}{r^2}\right) \tag{19}$$

II – Asset Prices

Consider a pure endowment economy, where the representative agent makes choices regarding consumption (C) and savings (B) solving the following problem:

$$\max_{\{C_t,B_t\}_{t=0}^{\infty}} \quad E_0\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\sigma}}{1-\sigma}$$

$$\text{s.t.} \quad Y_t + B_{t-1} \geq C_t + Q_tB_t,$$

where Q_t is the price of a risk-free bond delivering one unit of consumption in period $t+1$. The endowment Y_t is assumed to follow the exogenous process:

$$\frac{Y_t}{Y_{t-1}} = \varepsilon_t,$$

where $\log\varepsilon_t \sim N(0,s^2)$.

1 – Derive the first order conditions of the agent's problem and the market clearing condition.

Denote $\beta^t\lambda_t$ the Lagrange multiplier of the budget constraint in period t. First order condition w.r.t. C_t is $C_t^{-\sigma}\lambda_t$, FOC w.r.t. B_t gives $\lambda_tQ_t = \beta E_t\lambda_{t+1}$.
Good market equilibrium is $Y_t = C_t$ and bond market equilibrium is $B_t = 0$.

2 – Define a competitive equilibrium of this economy. Find an expression for the equilibrium value of Q_t that involves β, Y_t, Y_{t+1}, σ and the conditional expectation operator. (call this equation (1))

A competitive equilibrium is a sequence of quantities C_t, B_t and prices Q_t such that (i) FOC and budget constraint of the household are satisfied and (ii) markets clear.
We easily obtain

$$Q_t = \beta E_t\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \tag{1}$$

3 – What is the value of Q in a perfect foresight (deterministic) steady state?

$$Q_t = \beta \tag{2}$$

4 – Solve equation (1) to find an exact expression for the bond price Q_t as a function of constants and/or variables that are observable at time t or earlier. (You may need to use the following relationship: if X follows a log-normal distribution and $x = \log X$, then $E(X) = Ee^x = e^{E(x)+\frac{1}{2}Var(x)}$).

From the Euler equation,

$$Q_t = \beta E_t \left[\exp \left\{ -\sigma \log(\underbrace{Y_{t+1}/Y_t}_{\varepsilon_{t+1}}) \right\} \right]$$

Using the property of the log-normal distribution:

$$Q_t = \beta \exp \left\{ -\sigma E_t \left[\underbrace{\log \varepsilon_{t+1}}_{=0} \right] + \frac{\sigma^2}{2} \underbrace{Var(\log \varepsilon_{t+1})}_{=s^2} \right\}.$$

Therefore

$$Q_t = \beta \exp \left\{ \frac{\sigma^2 s^2}{2} \right\} \tag{3}$$

5 – What is the average value of Q? Is it the same as the steady state value computed in question **3**? Why or why not?

As seen in equation (3), Q_t is constant, and its average value is therefore $\beta \exp\left\{\frac{\sigma^2 s^2}{2}\right\}$. When there is risk in the economy ($s > 0$) and agents are risk averse ($\sigma > 0$), the price of the risk free asset is higher.

6 – How does the variance of the endowment process (s^2) affect the bond price? Why?

Observe equilibrium price equation (3). As the uncertainty s^2 increases, the bond price increases. This comes from precautionary savings motive. As the economy becomes more volatile, households are more willing to buy bonds in order to prepare for the uncertain future. In order to clear the market, the price of the bond should become higher. (Note that $u'''(c) > 0$ is required to have precautionary savings motive and the given utility function satisfies this requirement)

7 – Find the equilibrium solution for the interest rate $i_t = -\log Q_t$.

Take log to the both sides of the equation (3) and multiply by -1 yields

$$i_t = -\log Q_t = -\log \beta - \frac{\sigma^2 s^2}{2} \tag{4}$$

8 – Suppose that neither you nor your computer could find the exact solution obtained in the question above. Log-linearizing equation (1) around the steady-state, and obtaining an approximate solution for the interest rate might be an easier task. How good is such an approximation? To answer that question, show first that the Euler equation of the household can be written

$$E_t \left[\beta Q^{-1} \exp\left\{ -\widehat{q}_t - \sigma(\widehat{y}_{t+1} - \widehat{y}_t) \right\} \right] - 1 = 0 \tag{$\star$}$$

where for a generic variable x_t we have used the definition $\widehat{x}_t = \log X_t - \log X$. Then linearize (– *i.e.* take a first order Taylor expansion) the equation ($\star$) and use it to obtain an approximate solution for the interest rate as a function of constants and/or variables which are observable at time t or earlier. Show how to obtain the expression $-\widehat{q}_t - \sigma E_t \left[\widehat{y}_{t+1} - \widehat{y}_t\right] = 0$. How does the interest rate depend on the variance of the endowment process according to this approximate relation?

Rearranging the Euler equation we get

$$E_t \beta Q_t^{-1} \left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} - 1 = 0$$

which can be rewritten as

$$E_t \left[\beta Q^{-1} \exp\left\{ -\widehat{q}_t - \sigma(\widehat{y}_{t+1} - \widehat{y}_t) \right\} \right] - 1 = 0$$

where for a generic variable x_t we have used the definition $\widehat{x}_t = \log X_t - \log X$. (note that $\widehat{q}_t = 0$). Taking a first-order (Taylor) approximation to the above expression around the steady state, we obtain

$$-\widehat{q}_t - \sigma \underbrace{E_t \left[\widehat{y}_{t+1} - \widehat{y}_t\right]}_{=E_t \log \varepsilon_{t+1}=0} = 0;$$

Since $.\widehat{q}_t = \log Q_t - \log Q = -i_t - \log \beta$, we obtain

$$i_t = -\log \beta \tag{5}$$

Notice that (4) and (5) are different. This comes from the fact that log-linearization loses the information related to the moments of order t2 and above.

9 – Assume that the standard deviation of the endowment process (denoted s) can take values in the interval $[0.01, 0.1]$. Comparing the expressions obtained in questions **7** and **8**, calculate the approximation error (in percentage points), first assuming that the agent is risk-neutral ($\sigma = 0$) and then assuming that the agent has a degree of risk-aversion $\sigma = 5$. Discuss

Given $s \in [.01, .1]$, one has $s^2 \in [.0001, .01]$. There are two cases.

Case 1, $\sigma = 0$: linear utility, risk-neutral agents
Then (4) and (5) become identical so that in both of the cases,

$$i_t = \log \beta$$

This is the case because risk-neutral households only consider mean(first moment) for their decision. Thus, if households are risk-neutral, there is no approximation error.

Case 2, $\sigma = 5$: non-linear utility, risk-neutral agents
Using the given values and the equation (4), one can see that if we do not log-linearize, the equilibrium interest rate should be:

$$i_t \in [-\log \beta - .125, -\log \beta - .00125]$$

while $i_t = -\log \beta$ when we consider the linearized model. As a result, the approximation error is from .125% to 12.5%. Thus as increase of uncertainty (s^2), the approximation error also increases. In summary, log-linearization overestimates the interest rate (equivalently, underestimate the price of the bond) because it ignores the presence of uncertainty, and the associated precautionary savings motive, which is a driving force to drive down the interest rate (or increase the price of the bond).