AUTUMN TERM MRES ASSESSMENT 2024/25
EXAMINATION COVER SHEET

Module Code:	**ECON0105**
Module Title:	**Macroeconomics**
ASSESSMENT COMPONENT:	**40% EXAM**
ASSESSMENT DATE:	**15th January 2025**
ASSESSMENT START TIME:	**13:30pm**
ASSESSMENT END TIME:	**16:30pm.**
TIME ALLOWED:	**3 hours, closed book.**

INSTRUCTIONS

Write in **blue** or **black ink.**

On the first page of your answer booklet be sure to provide:

- your candidate number
- the number of booklets you submitted.

On each of your answer booklets, make sure to write:

- your candidate number
- the module code

Do not write your name anywhere on the answer sheets;

Do not begin reading the question paper before the start of the exam;

Do check both sides of the exam paper for questions;

Complete all rough work in your answer sheets and cross through any work you do not want marked;

Secure your answer booklets with a treasury tag.

Return the exam paper and your answer booklet to the invigilators. Do not take any of the documents with you when leaving the room.

Raise your hand to alert the invigilator if you require more answer paper, feel unwell or need to take a break (for example to go to the toilet).

University College London, 2024-2025
Econ 0107 – Macroeconomics I – Łukasz Rachel & Franck Portier

Midterm Assessment

Answer the four problems. I and II: 35% of the total mark each. III and IV: 15% of the total mark each.

I – Sustainability of government debt when $R < G$

Consider an economy populated with a continuum of households, a representative firm and a government. We carry our analysis in a detrended model, where each variable that we introduce is already normalized by the balanced growth path level of output. This will become explicit and clear in a moment.

Each household supplies one unit of labour inelastically, and solves the following problem

$$\max_{\{c_t,b_t\}_{t=0}^{\infty}} \int_{t=0}^{\infty} e^{-\rho t}(\log c_t + v(b_t))dt \quad \text{subject to} \\ c_t + \dot{b}_t \leq (R_t - G)b_t + w_t + d_t \tag{P}$$

where ρ is the discount rate, c_t is consumption (relative to balanced growth path level of output), b_t is government debt holdings (also relative to balanced growth path level of output $b_t := \frac{B_t}{Y_t}$), $v(b_t)$ is utility from holding government debt satisfying $v' \geq 0,\ v'' \leq 0$, w_t is the detrended wage rate and d_t is detrended lump sum transfers (or, if negative, taxes).

Competitive representative firm produces using constant returns technology $Y_t = A_t \cdot n_t$, where A_t grows at a constant exponential rate γ and n_t is the labor input. Note that labor market clearing condition implies $n_t = 1 \forall t$. Therefore $Y_t = A_t$, and detrended output is $y_t = 1 \forall t$.

The government sets fiscal and monetary policy. Fiscal policy consists of paths $\{b_t, d_t\}$ subject to the flow budget constraint

$$(R_t - G)b_t + d_t = \dot{b}_t$$

where R_t is the net nominal interest rate, G is the net nominal growth rate of output and d_t is the primary deficit. Monetary policy sets the interest rate equal to the steady state (natural) rate of interest, which results in inflation being equal to the inflation target, $\pi = \pi^*$. Thus, we have $G = \gamma + \pi^*$.

1 – Write the Hamilton-Jacobi-Bellman equation corresponding to problem (P). Take the first order conditions. Provide the economic intuition for each of the conditions. Derive the household Euler Equation.

Solution:

$$\rho V(b) = \max_{c,b} \log c + v(b) + V_b(b)\left((R_t - G_t)b_t + w_t - d_t - c_t\right)$$

The FOC is:

$$\frac{1}{c} = V_b$$

which says that at the optimum the marginal utility of consuming a dollar of wealth equals the marginal value of the dolar.
The Envelope condition is:

$$\rho V_b = v_b + V_b(R_t - G_t) + V_{bb}\dot{b}$$

which says that the required return is compensated by the flow pay-off coming from utility of holding wealth and earning interest on it, plus the change in the value of wealth over time.
Differentiating the FOC wrt time, we obtain:

$$-\frac{\dot{c}}{c} = \frac{V_{bb}}{V_b}\dot{b}$$

Thus:

$$\frac{\dot{c}}{c} = (R_t - G_t) - \rho + v_b c$$

2 – Comment on what happens to an individual agent's consumption profile over time when the impatience motive is exactly offset by the financial incentive to save (provided by interest rate net of the growth rate).

Solution: When $\rho - (R_t - G_t) = 0$, $\frac{\dot{c}}{c}$ is positive. The fact that bond feature directly in utility provides an additional incentive to save, over and above the financial incentive.

3 – Combine the government budget constraint and the households' budget constraint to show that the two imply that the resource constraint is satisfied.

Solution: [1 MARK]

$$c_t = w_t = A_t/Y_t = 1$$

4 – For a steady state level of debt b, derive the expression for a natural rate of interest $R^*(b)$. This is the interest rate that holds in a steady state / on the balanced growth path in this economy. Explain the intuition behind this result, both in terms of the level of the natural real interest rate as well as how it changes with b. Can the nominal interest rate be negative?

Solution:

$$R(b) = G + \rho - v_b.$$

Since $v' \geq 0$, $v'' \leq 0$, the steady state interest rate is lower than in the neoclassical growth model, reflecting an additional motive to save. The interest rate increases with government debt b: if the government issues a lot of debt, the convenience premium shrinks. The nominal rate can of course be negative, if b is sufficiently small.

5 – Using the government budget constraint and the result for $R(b)$, compute the level of primary deficit d that keeps debt constant. Compute the level of debt b^* for which $d(b)$ is maximized.

Solution:

$$d(b) = (G - R(b))b = (v_b - \rho)b.$$

This is maximized at $b^* = \frac{\rho - v_b}{v_{bb}}$.

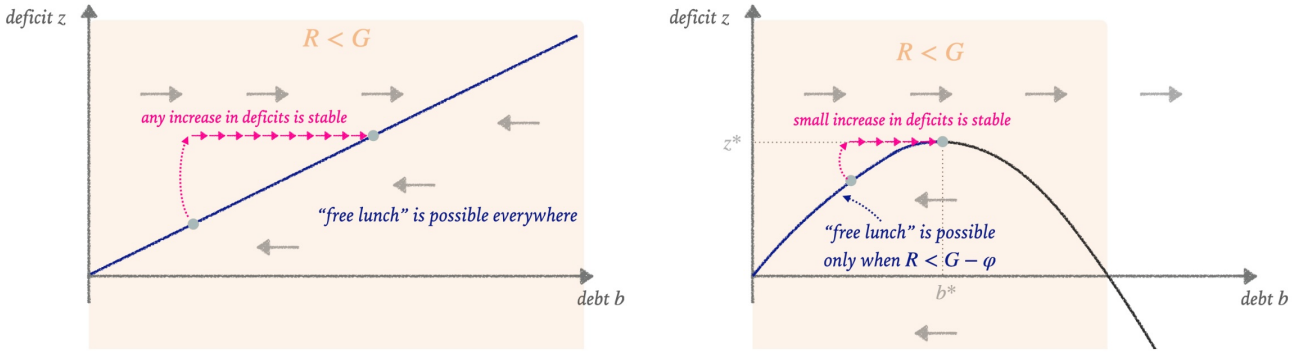
6 – Now consider for a moment the government budget constraint with constant and exogenous G and R. Draw two phase diagrams, both with d on the vertical and b on the horizontal axis, the left panel assuming $R > G$ and the right panel assuming $R < G$. Make sure to draw in the arrows for the dynamics of b in both cases. Using your analysis above, explain how, starting in a steady state, a permanent increase in government deficit can be sustained indefinitely when $R < G$, in the sense that it results in a stable level of debt (even without any fiscal consolidation).

Solution: Recall $(R - G)b_t + d_t = \dot{b}_t$. Thus $\dot{b} = 0$ implies $d = -(R - G)b$. We only plot the case with $R < G$. See the left panel of the figure, which is from Mian, Straub and Sufi's (2024) Goldilocks paper.

7 – Returning to our model, and assuming that $v_b(0) > \rho$ (and thus $R < G$ for b not too large) and $v_b(0) < \infty$, draw the phase diagram. Start by plotting $d(b)$ you derived above, with b on the horizontal and d on the vertical axis. Make sure to draw the arrows indicating debt dynamics. Show (using your diagram) that, starting at some steady state, the government can increase primary deficit and never pay for it (and still stabilize debt in the long-run, albeit at a higher level) as long as the level of debt remains below b^* you derived above. Using that

Figure 1: Debt-deficit dynamics

(a) Model with exogenous $R < G$ **(b)** Model with endogenous $R^*(b)$



insight, show that such *free lunch* is possible when

$$R(b) < G + bv_{bb} < G.$$

Interpret this finding – in particular, what are the fiscal policy implications of realizing that the natural interest rate is endogenous to the quantity of debt issued?

Solution: See the right panel in the figure above. Recall $(R(b) - G)b_t + d_t = \dot{b}_t$. Thus $\dot{b} = 0$ implies $d = -(R(b) - G)b = (v_b - \rho)b$. Clearly this schedule is hump-shaped and goes through the origin (since we assumed $v_b(0) < \infty$).

8 – Suppose that, under political pressure, the central bank raises the inflation target to $\pi^{**} > \pi^*$. Illustrate how this change affects the government debt sustainability analysis in your phase diagram. Does raising the inflation target help the government?

Solution: No, this raises G and R by the same amount (households have a smaller desire to save in the face of higher inflation), and does not affect b^*.

II – Two Agents with Complete Markets

A pure endowment economy consists of two types of consumers. Consumers of type a order consumption streams of the one good according to

$$\sum_{t=0}^{\infty} \beta^t c_t^a.$$

Note that instantaneous utility is linear in c, so that we will have corner solutions (c_t^a can be zero).

Consumers of type b order consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^b),$$

$c_t^i \geq 0$ is the consumption of a type i consumer and $\beta \in (0, 1)$ is a common discount factor. The consumption good is tradable but nonstorable. There are equal numbers (one) of the two types of consumers.

The consumer of type a receives the following endowment:

$$y_t^a = \mu \quad \forall t \geq 0, \quad \mu > 0,$$

and the consumer of type b receives:

$$y_t^b = \begin{cases} 0 & \text{if } t \geq 0 \text{ is even,} \\ \alpha & \text{if } t \geq 0 \text{ is odd,} \end{cases}$$

where

$$\alpha = \mu \frac{(1+\beta)}{\beta}. \tag{$\mathcal{C}$}$$

Time is discrete and goes from 0 to ∞.

1 – Define a competitive equilibrium with time-0 trading. Be careful to include definitions of all of the objects of which a competitive equilibrium is composed.

Solution: A competitive equilibrium with time-0 trading is a feasible allocation $\{c_t^a, c_t^b\}_{t=0}^{\infty}$ (non negative) and price system $\{q_t^0\}_{t=0}^{\infty}$ (strictly positive) such that, given the price system, the allocation solves each consumer's problem.

2 – Compute a competitive equilibrium allocation with time-0 trading (prices $\{q_t^0\}$ and quantities $\{c_t^a, c_t^b\}$). It is useful to start guessing that c_t^b will be a constant c^b (to be determined) (explain why this is a good guess). From that guess, compute equilibrium prices and quantities starting from the Euler equation of agent b and check that the guess is indeed a correct guess for the obtained equilibrium prices. You can use the normalisation $q_0^0 = 1$. Discuss the results.

Solution: Consider type b consumers. As their marginal utility is strictly positive, the constraint $c_t^b \geq 0$ will never be binding and their intertemporal budget constraint will always bind. Their problem can there be written:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t^b + \lambda^b \left(\sum_{t=0}^{\infty} q_t^0 \left(y_t^b - c_t^b \right) \right)$$

and first order conditions are

$$\frac{\beta^t}{c_t^b} = \lambda^b q_t^0, \tag{1}$$

$$\lambda^b \left(\sum_{t=0}^{\infty} q_t^0 \left(y_t^b - c_t^b \right) \right) = 0, \tag{2}$$

$$\lambda^b \geq 0. \tag{3}$$

Taking the ratio of (**??**) for t and $t+1$, we obtain the Euler equation:

$$c_{t+1}^b = \beta \frac{q_t^0}{q_{t+1}^0} c_t^b.$$

Guessing that $c_t^b = c^b \ \forall \ t$ and with the normalisation $q_0^0 = 1$, we obtain

$$q_t^0 = \beta^t.$$

By construction, a constant c_t^b is optimal for the consumer b for this set of prices. We can find the value of c^b from their intertemporal budget constraint (ibc):

$$\sum_{t=0}^{\infty} q_t^0 y_t^b = \sum_{t=0}^{\infty} q_t^0 c_t^b,$$

which rewrites

$$\sum_{t \text{ odd}} \beta^t \alpha = \sum_{t=0}^{\infty} q_t^0 y_t^b c^b,$$

or

$$\frac{\alpha\beta}{1-\beta^2} = \frac{c^b}{1-\beta},$$

and therefore

$$c^b = \frac{\alpha\beta}{1+\beta}.$$

Using the feasibility condition $c_t^a + c_t^b = y_t^a + y_t^b$, we obtain:

$$c_t^a = \begin{cases} \mu - \frac{\alpha\beta}{1+\beta} & = \mu - \frac{\alpha\beta}{1+\beta} \quad \text{if } t \text{ even}, \\ \mu + \alpha - \frac{\alpha\beta}{1+\beta} & = \mu + \frac{\alpha}{1+\beta} \quad \text{if } t \text{ odd}. \end{cases}$$

Last thing we need to check is that c_t^a is always positive, which is satisfied if $c^a_{\text{even}} \geq 0$. Using the condition $\alpha = \mu\frac{(1+\beta)}{\beta}$, we find $c^a_{\text{even}} = 0$.

We see that in equilibrium the agent with infinite intertemporal elasticity of substitution (or risk neutral agent – consumer b) accepts fluctuation in their consumption to allow consumer a (the risk averse one) to smooth consumption. There are gains fro trade between the two agents because they do not have same preferences nor endowment streams.

3 – Compute the sequence of one-period gross interest rates between t and $t+1$, denoted R_t. Do they differ between odd and even periods?

Solution:

$$R_t = \frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}.$$

The interest rate is constant (otherwise consumer a would not choose a constant consumption stream).

$\star$

Now consider an economy identical to the preceding one except in one respect. The endowment of consumer a continues to be μ each period, but we assume that the endowment of consumer b is larger (though it continues to be zero in every even period). In particular, we alter the assumption about endowments in condition $(\mathcal{C})$ to the new condition:

$$\alpha > \mu\frac{(1+\beta)}{\beta}. \tag{$\mathcal{C}'$}$$

We will look for a competitive equilibrium of this economy in which $c^a_{t \text{ is even}} = c^a_{\text{even}} = 0$ and $c^a_{t \text{ is odd}} = c^a_{\text{odd}} > 0 \ \ \forall t$.

4 – Using the optimal behaviour of consumer a, explain why in this equilibrium we must have

$$q_t^0 = \beta^{t-\tau} q_\tau^0 \text{ for } t > \tau \text{ and } t, \tau \text{ odd}.$$

or equivalently

$$q_t^0 = \beta^{t-1} q_1^0 \quad \text{for } t \text{ odd}. \tag{$\star$}$$

Solution: If (say) $q_t^0 > \beta^{t-\tau} q_\tau^0$ for $t > \tau$ and t, τ odd, then consumer a (with linear utility) have an incentive to shift consumption from t to τ. As marginal utility of consumption is constant, it would be optimal to do this up to $c_t^a = 0$, which violates the assumption $c^a_{\text{odd}} > 0$. Same argument can be made when $q_t^0 < \beta^{t-\tau} q_\tau^0$. Therefore, on must have in this equilibrium $q_t^0 = \beta^{t-\tau} q_\tau^0$ for $t > \tau$ and t, τ odd.

5 – Find the competitive equilibrium allocation and price system for this economy (use $(\star)$, the Euler equation of consumer b between period t even and period 0 and the ibc of consumer b to find c^b_{odd}, and from there prices and quantities).

Solution: From the feasibility condition in even periods, we have $c^b_{\text{even}} = \mu$. The Euler equation of consumer b between period t even and period 1 (which is odd) then writes:

$$q^0_t = \beta^{t-1} q^0_1 \frac{c^b_{\text{odd}}}{\mu} \qquad (\star\star)$$

Then, using $(\star)$ and $(\star\star)$, we obtain

$$\sum_{t=0}^{\infty} q^0_t y^b_t = \sum_{t \text{ odd}}^{\infty} q^0_t \alpha + \sum_{t \text{ even}}^{\infty} q^0_t \times 0 = \sum_{t \text{ odd}}^{\infty} \beta^{t-1} q^0_1 \alpha = \frac{\alpha q^0_1}{1-\beta^2}.$$

On the expenditures side,

$$\sum_{t=0}^{\infty} q^0_t c^b_t = \sum_{t \text{ odd}}^{\infty} q^0_t c^b_{\text{odd}} + \sum_{t \text{ even}}^{\infty} q^0_t \mu = \sum_{t \text{ odd}}^{\infty} \beta^{t-1} q^0_1 c^b_{\text{odd}} + \sum_{t \text{ even}}^{\infty} \beta^{t-1} q^0_1 \frac{c^b_{\text{odd}}}{\mu} \mu = q^0_1 c^b_{\text{odd}} \frac{1}{1-\beta^2} \frac{1+\beta}{\beta}.$$

Along the agent b ibc, we then have:

$$\frac{\alpha q^0_1}{1-\beta^2} = q^0_1 c^b_{\text{odd}} \frac{1}{1-\beta^2} \frac{1+\beta}{\beta},$$

which gives

$$c^b_{\text{odd}} = \frac{\alpha\beta}{1+\beta}.$$

We then need to check that $c^a_{\text{odd}} = \mu - c^b_{\text{odd}} > 0$, which is satisfied under condition $(\mathcal{C}')$.

Finally, we need to find q^0_1. Take the Euler equation of agent b between 0 and 1:

$$\underbrace{q^0_0}_{=1} = \beta^{-1} q^0_1 \frac{\left(\frac{\alpha\beta}{1+\beta}\right)}{\mu},$$

which gives

$$q^0_1 = \frac{\mu(1+\beta)}{\alpha},$$

so that

$$q^0_t = \begin{cases} \beta^t \frac{\alpha}{(1+\beta)\mu} q^0_1 & \text{when } t \text{ even,} \\ \beta^t \frac{1}{\beta} q^0_1 & \text{when } t \text{ odd.} \end{cases}$$

6 – Compute the sequence of one-period interest rates implicit in the equilibrium price system that you computed in question **5**. Are interest rates higher or lower than those you computed in question **2**?

Solution:

$$R_t = \frac{q^0_t}{q^0_{t+1}} = \begin{cases} \frac{1}{\beta} \frac{\alpha\beta}{(1+\beta)\mu} & \text{when } t \text{ even,} \\ \frac{1}{\beta} \frac{(1+\beta)\mu}{\alpha\beta} & \text{when } t \text{ odd.} \end{cases}$$

The interest rate alternates between high (higher than in question **2**) and low (lower than in question **2**). In even periods, R is is high, which incentivises consumer a to save, as it is a period in which consumer b needs to borrow to get positive consumption (as their endowment is zero).

III – Aiyagari and Inequality

1 – Write down the Bellman equation of a consumer in the Bewley-Hugget-Aiyagari type of model with a two state income process and no aggregate uncertainty.

2 – Explain in words why in the Aiyagari model, if there is aggregate uncertainty, the wealth distribution is one of the relevant state variables.

3 – When calibrated in a standard fashion, the Aiyagari model fails to replicate the large amount of wealth concentration at the top of the wealth distribution in the United States, United Kingdom and other developed countries. In particular it fails to match statistics like the top 1% and top 0.1% wealth shares and the observed Pareto tail of empirical wealth distributions. List three extensions of the Aiyagari model that researchers have proposed to explain top wealth inequality, and in each case provide a one-sentence intuition for the proposed mechanism.

Solution: The Bellman equation is standard, as is the discussion of the aggregate uncertainty case (agents need to know the distribution to forecast prices).

- Heterogeneity in discount factors: patients households (richer) save more (Krusell and Smith, 1998)
- Non-homothetic preferences: rich save more, e.g. bequests (Atkinson, De Nardi, Straub)
- Super high but transitory income realization – "awesome state": rich save more for precautionary reasons (Castaneda et al, 2003)
- Heterogeneous/stochastic rates of return (Benhabib & al, 2014)
- Entrepreneurs with projects yielding higher, but stochastic, rate of return (Quadrini, 2000)

IV – Optimal Taxation with Commitment

Consider the environment of Lecture 5 of Franck Portier (infinite horizon, deterministic environment, complete markets, one representative household and one representative firm with Neoclassical production function and labour and capital as inputs, a gouvernment that spends an exogenous stream and taxes using labour income tax, capital income tax and issues one period real debt).

1 – Explain the logic of the primal and dual approach to fin the optimal path of taxes.

Solution: In an optimal taxation with commitment problem, The government's goal is to maximize households' welfare subject to raising revenues through distortionary taxation. When designing an optimal policy, the government takes into account the competitive equilibrium reactions by consumers and firms to the tax system. This problem is referred to as a *Ramsey problem*. By commitment, we mean that the gouvernement chooses the path of taxes in period 0 and is not allowed to change them afterwards.

In the dual approach, we solve assuming that the government chooses taxes given that for an announced path of taxes, the private sector will play a competitive equilibrium.

In the primal approach, we solve by first using the equations defining the competitive equilibrium to eliminate all prices, and collapse the competitive equilibrium restriction on quantities into a so-called "implementability condition". Then we solve for the optimal quantity choices under resource constraints and the "implementability condition". Then taxes and prices can be backed out from quantities.

None of the two approaches is "better" than the other. It is more a matter of convenience and interpretation.

2 – Explain the result of zero capital tax at the steady state.

Solution: The result of zero capital tax in the steady state can be closely related to the 1971 Diamond-Mirrlees production efficiency result in optimal taxation.

The Diamond-Mirrlees result establishes that in an optimal taxation system, it is optimal to ensure production efficiency, meaning that production should occur on the production possibility frontier. There should be no distortions in the allocation of inputs between sectors or firms. Therefore, the government should rely on taxes that distort consumption and labor decisions rather than input choices in production.

The zero capital tax at the steady state result can be viewed as an application of the Diamond-Mirrlees principle of production efficiency in a dynamic setting. In the growth model framework, capital is an input that is produced (as opposed to labour) Taxing capital income distorts the accumulation of capital over time, leading to suboptimal levels of capital stock in the steady state. This violates the principle of production efficiency highlighted by Diamond-Mirrlees because the distortion affects the marginal productivity of capital and leads to inefficiency in production.