2024-2025 – Econ 0107 – Macroeconomics I

Lecture 5 : Optimal Taxation with Commitment

(Chapter 16 in Ljungqvist & Sargent)

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1. Introduction

- We study a dynamic optimal taxation problem with commitment = Ramsey *problem*
- The solution to this problem is referred to as a Ramsey *plan*
- The government's objective goal is to maximize households welfare subject to raising revenues through distortionary taxation.
- When designing an optimal policy, the government takes into account the competitive equilibrium reactions by consumers and firms to the tax system.
- Commitment: All taxes are determined in period 0 and the govt commits.
- What if not? Time inconsistency of capital taxation.

2. The economy

- No uncertainty
- Representative household (hh):

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t) \tag{2.1}$$

- Technology

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1 - \delta)k_t \tag{2.3}$$

- F linearly homogenous of degree 1
- Govt:
  - × spends exogenously $\{g_t\}$
  - × finances with $\{\tau_{kt}\}$,$\{\tau_{nt}\}$ and one period real bonds $\{b_t\}$
  - × Budget constraint

$$g_t = \tau_{kt} r_t k_t + \tau_{nt} w_t n_t + \frac{b_{t+1}}{R_t} - b_t \quad , \quad \lim_{T \to \infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) b_{T+1} = 0 \tag{2.5}$$

  - × Bonds are taxed exempted, no depreciation allowance

2.2. Households

- Choose $\{c_t, n_t, k_{t+1}, b_{t+1}\}$ to max

$$\begin{aligned}\max \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \Big(U(c_t, 1-n_t) \\ &+ \lambda_t \big((1-\tau_{nt}) w_t n_t + (1-\tau_{kt}) r_t k_t + (1-\delta) k_t + b_t - c_t - k_{t+1} - \frac{b_{t+1}}{R_t} \big) \Big)\end{aligned}$$

with k_0, b_0 given. *focs* are:

$$\begin{aligned} U_{1t} &= \lambda_t && (2.7) \\ U_{2t} &= \lambda_t (1-\tau_{nt}) w_t && (2.8) \\ \lambda_t &= \beta \lambda_{t+1} ((1-\tau_{kt+1}) r_{t+1} + 1 - \delta) && (2.9) \\ \lambda_t &= \beta \lambda_{t+1} R_t && (2.10) \end{aligned}$$

- which implies

$$\begin{aligned} U_{2t} &= U_{1t} (1-\tau_{nt}) w_t && (2.11a) \\ U_{1t} &= \beta U_{1t+1} ((1-\tau_{kt+1}) r_{t+1} + 1 - \delta) && (2.11b) \end{aligned}$$

- No-arbitrage condition: $R_t = (1-\tau_{kt+1}) r_{t+1} + 1 - \delta$ (2.12)

- We also have the transversality conditions

$$\lim_{T\to\infty}\left(\prod_{i=0}^{T-1} R_i^{-1}\right) k_{T+1} = 0 \tag{2.15}$$

$$\lim_{T\to\infty}\left(\prod_{i=0}^{T-1} R_i^{-1}\right) \frac{b_{T+1}}{R_T} = 0 \tag{2.16}$$

2.3. Representative firm

- Chooses $\{n_t, k_t\}$ to max $\{\Pi_t = F(k_t, n_t) - r_t k_t - w_t n_t\}$
- No-arbitrage:

$$\begin{aligned} r_t &= F_{kt} \\ w_t &= F_{nt} \end{aligned}$$

3. Ramsey problem

Definition 1

A feasible allocation is $\{k_t, c_t, n_t, g_t\}$ *that satisfies*

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1 - \delta)k_t \tag{2.3}$$

Definition 2

A price system is a nonnegative bounded sequence $\{w_t, r_t, R_t\}$

Definition 3

A government policy is a sequence $\{g_t, \tau_{kt}, \tau_{nt}, b_t\}$

Definition 4

A competitive equilibrium is a feasible allocation, a price system and a govt policy such that

1. *given price system and govt policy, allocation solves hh and firm problem*
2. *given price system and allocation, govt policy satisfies the sequence of budget constraints and transversality condition:*

$$g_t = \tau_{kt} r_t k_t + \tau_{nt} w_t n_t + \frac{b_{t+1}}{R_t} - b_t \quad , \quad \lim_{T \to \infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) b_{T+1} = 0 \qquad (2.5)$$

3. RAMSEY problem

Definition 5

Given k_0, b_0 and $\{g_t\}$, the RAMSEY *problem is to choose $\{\tau_{kt}, \tau_{nt}\}$ so that the corresponding competitive equilibrium maximises utility.*

- We will impose the restriction that τ_{k0} is given and "small"
- Why? Because a trivial solution to the RAMSEY problem is to use only τ_{k0} to finance $\{g_t\}$.
- Why? k_0 is given $\rightsquigarrow$ τ_{k0} is not distortive. (more later)

4. Zero capital tax at the steady state

- Assume (without loss of generality) that the govt chooses $\widetilde{r}_t = (1 - \tau_{kt})r_t$ and $\widetilde{w}_t = (1 - \tau_{nt})n_t$ instead of τ_{kt} and τ_{nt}.
- Tax revenues are

$$\begin{aligned} \tau_{kt} r_t k_t + \tau_{nt} w_t n_t &= (r_t - \widetilde{r}_t)k_t + (w_t - \widetilde{w}_t)n_t \\ &= r_t k_t + w_t n_t - \widetilde{r}_t k_t - \widetilde{w}_t n_t \\ &= \underbrace{F_{kt} k_t + F_{nt} n_t}_{F(k_t, n_t)} - \widetilde{r}_t k_t - \widetilde{w}_t n_t \end{aligned}$$

- Plug this expression in the govt budget constraint:

$$F(k_t, n_t) - \widetilde{r}_t k_t - \widetilde{w}_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t = 0$$

- This equation collapses the govt *bc* and the firm's *focs*

4. Zero capital tax at the steady state

- The other constraints the govt is facing when choosing optimal taxes are :
  - × Resource constraint:

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta)k_t$$

  - × Hh *focs*:

$$\begin{aligned} U_{2t} &= U_{1t}(1-\tau_{nt})w_t \\ U_{1t} &= \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta) \end{aligned}$$

  - × No-arbitrage condition

$$R_t = (1-\tau_{kt+1})r_{t+1} + 1 - \delta$$

4. Zero capital tax at the steady state

- To derive the RAMSEY plan, the govt maximizes $\mathcal{L}$ by optimally choosing $\{k_{t+1}, c_t, n_t, \widetilde{w}_t, \widetilde{r}_t\}$

$$\begin{aligned}\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big[U(c_t, n_t) \quad &+ \quad \psi_t\big(F(k_t, n_t) - \widetilde{r}_t k_t - \widetilde{w}_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t\big) \\ &+ \quad \theta_t\big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1}\big) \\ &+ \quad \mu_{1t}\big(U_{2t} - U_{1t}(1-\tau_{nt})w_t\big) \\ &+ \quad \mu_{2t}\big(U_{1t} - \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta)\big)\Big]\end{aligned} \tag{4.1}$$

with $r_t = F_{kt}$ and $w_t = F_{nt}$.

4. Zero capital tax at the steady state

- The *foc* wrt k_{t+1} is

$$\underbrace{\theta_t}_{\substack{\text{cost of investing} \\ \text{one more unit}}} = \underbrace{\beta \left(\underbrace{\psi_{t+1}(F_{kt+1} - \widetilde{r}_{t+1})}_{\substack{\text{more tax revenues} \\ \text{tomorrow}}} + \underbrace{\theta_{t+1}(F_{kt+1} + 1 - \delta)}_{\substack{\text{more goods} \\ \text{tomorrow}}} \right)}_{\text{benefits of investing one more unit}} \tag{4.2}$$

- Assume that g_t stays constant after T so that $\exists$ a steady state to which the RAMSEY plan is converging.
- Using $r_t = F_{kt}$, (4.2) rewrites at the steady state:

$$\theta = \beta(\psi(r - \widetilde{r}) + \theta(r + 1 - \delta)) \tag{4.3}$$

4. Zero capital tax at the steady state

- Use the Hh Euler equation $U_{1t} = \beta U_{1t+1}((1-\tau_{kt+1})r_{t+1} + 1 - \delta)$ at the steady state:

$$1 = \beta(\widetilde{r} + 1 - \delta)$$

which implies

$$\begin{aligned} \beta(r + 1 - \delta) &= \beta(r - \widetilde{r} + \widetilde{r} + 1 - \delta) \\ &= \beta(r - \widetilde{r}) + \underbrace{\beta(\widetilde{r} + 1 - \delta)}_{1} \end{aligned}$$

- (4.3) becomes

$$\theta = \beta\psi(r - \widetilde{r}) + \theta \underbrace{\beta(r + 1 - \delta))}_{\beta(r-\widetilde{r})+1}$$

which implies

$$(\theta + \psi)(r - \widetilde{r}) = 0 \tag{4.5}$$

4. Zero capital tax at the steady state

$$(\theta + \psi)(r - \widetilde{r}) = 0 \tag{4.5}$$

Since $\theta > 0$ and $\psi \geq 0$, one must have

$$r = \widetilde{r}$$

which implies

$$\tau_k = 0$$

Chamley-Judd result of no taxation of capital at the steady state.

Proposition 1

If $\exists$ a steady state Ramsey *allocation, the associated limiting tax rate on capital income is zero.*

$\rightsquigarrow$ Diamond-Mirlees no-taxation of intermediate goods result (Diamond & Mirrlees, *Optimal Taxation and Public Production I: Production Efficiency.* American Economic Review, (1971)).

In a competitive economy with constant returns to scale (or decreasing returns to scale and a 100 percent tax on profits) commodity taxation should not distort the input choices of firms.

The policy implication of the lemma is that commodity taxes should be levied on final consumption goods and not on inputs to production.

This lemma justifies the use of a value-added tax system in which producers can reclaim the tax paid on inputs.

`A Dictionary of Economics (3 ed.), Black, Hashimzade and Myles, 2009`

6. Primal approach to the RAMSEY problem

- Previously we have eliminated (τ_k, τ_n) and (w_t, r_t) and replace with $(\widetilde{w}_t, \widetilde{r}_t)$
- Let's go further and eliminate all taxes and prices.
  - × Dual approach: tax rates are the govt decisions
  - × Primal approach: the govt directly chooses quantities under the constraints imposed by competitive equilibrium

6. Primal approach to the RAMSEY problem

- The dual problem is

$$\begin{aligned}\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big[U(c_t, n_t) \quad &+ \quad \psi_t \big(F(k_t, n_t) - \widetilde{r}_t k_t - \widetilde{w}_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t \big) \\ &+ \quad \theta_t \big(F(k_t, n_t) + (1-\delta) k_t - c_t - g_t - k_{t+1} \big) \\ &+ \quad \mu_{1t} \big(U_{2t} - U_{1t}(1-\tau_{nt}) w_t \big) \\ &+ \quad \mu_{2t} \big(U_{1t} - \beta U_{1t+1}((1-\tau_{kt+1}) r_{t+1} + 1 - \delta) \big) \Big]\end{aligned} \tag{4.1}$$

- Let's eliminate all prices and taxes
- Instead of the sequence of gvt budget constraints in blue, we will use the intertemporal budget constraint of the hh (this is equivalent by WALRAS law) + the no-arbitrage condition ($q_0^0 = 1$ (numéraire)):

$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} q_t^0 (1-\tau_{nt}) w_t n_t + ((1-\tau_k) r_0 + 1 - \delta) k_0 + b_0$$

6. Primal approach to the RAMSEY problem

Steps

Step 1 Obtain the *focs* of hh and firm + no-arbitrage pricing conditions and solve for $\{q_t^0, r_t, w_t, \tau_{kt}, \tau_{nt}\}$ as a function of $\{c_t, n_t, k_{t+1}\}$

Step 2 Substitute these expressions into the intertemporal *bc* if the hh $\rightsquigarrow$no more prices or taxes in that equation = *implementability condition*.

Step 3 Solve the RAMSEY problem by maximizing welfare s.t. resource constraint and implementability condition.

Step 4 Once allocations are obtained, use the formulas of step 1 to obtain taxes.

6. Primal approach to the RAMSEY problem

Step 1

- Hh problem:

$$\max \sum \beta^t U(c_t, 1-n_t) + \lambda \left(-\sum_{t=0}^{\infty} q_t^0 c_t + \sum_{t=0}^{\infty} q_t^0 (1-\tau_{nt}) w_t n_t + ((1-\tau_k) r_0 + 1 - \delta) k_0 + b_0 \right)$$

- *focs* are

$$\begin{aligned} \beta^t U_{1t} &= \lambda q_t^0 \\ \beta^t U_{2t} &= \lambda q_t^0 (1-\tau_{nt}) w_t \end{aligned}$$

- with $q_0^0 = 1$ (numéraire), we have

$$\begin{aligned} q_t^0 &= \beta^t \frac{U_{1t}}{U_{1,0}} \quad (6.3a) \\ (1-\tau_{nt}) w_t &= \frac{U_{2t}}{U_{1t}} \quad (6.3b) \end{aligned}$$

- Note that now taxes and prices are functions of quantities.

- Plus no-arbitrage conditions

$$\begin{aligned} \frac{q_t^0}{q_{t+1}^0} &= (1-\tau_{kt+1})r_{t+1}+1-\delta \\ r_t &= F_{kt} \\ w_t &= F_{nt} \end{aligned}$$

- Put (6.3a,b) and $r_0 = F_{k0}$ in the *ibc* of the hh

$$\sum_{t=0}^{\infty} \underbrace{q_t^0}_{\beta^t \frac{U_{1t}}{U_{1,0}}} c_t = \sum_{t=0}^{\infty} \underbrace{q_t^0}_{\beta^t \frac{U_{1t}}{U_{1,0}}} \underbrace{(1-\tau_{nt}) w_t}_{\frac{U_{2t}}{U_{1t}}} n_t + \Big((1-\tau_{k0}) \underbrace{r_0}_{F_{k0}} + 1 - \delta \Big) k_0 + b_0$$

which gives

$$\sum_{t=0}^{\infty} \beta^t (U_{1t} c_t - U_{2t} n_t) - A_0 = 0 \tag{6.5}$$

with

$$A_0 = A_0(c_0, n_0, \tau_0^k, b_0) = U_{1,0} \Big(((1-\tau_{k0}) F_{k0} k_0 + 1 - \delta) k_0 + b_0 \Big) \tag{6.6}$$

- Let Φ be the Lagrange multiplier to the hh *ibc* (6.5)
- Define
$$V(c_t, n_t, \Phi) = U(c_t, 1 - n_t) + \Phi(U_{1t} c_t - U_{2t} n_t)$$
- Write the Lagrangian
$$\mathcal{J} = \sum_{t=0}^{\infty} \beta^t \Big(V(c_t, n_t, \Phi) + \theta_t \big(F(k_t, n_t) + (1 - \delta) k_t - c_t - g_t - k_{t+1} \big) \Big) - \Phi A_0$$
- The govt problem is then to max $\mathcal{J}$ wrt $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ for given $k_0, b_0, \tau_{k0} \rightsquigarrow$ only a choice of quantities in this problem.

- Interpretation: In that primal problem, the govt is choosing quantities under the constraint that these quantities can be decentralized with an appropriate set of optimal taxed (this is the *implementability condition*)
- Φ = utility cost of raising resources through distortionary taxes

$$\max \mathcal{J} = \sum_{t=0}^{\infty} \beta^t \Big(V(c_t, n_t, \Phi) + \theta_t \big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1} \big) \Big) - \Phi A_0$$

focs are:

$$\begin{array}{lllll}
c_t: & V_{ct} & = & \theta_t & \forall t \geq 1 \\
n_t: & V_{nt} & = & -\theta_t F_{nt} & \forall t \geq 1 \\
k_{t+1}: & \theta_t & = & \beta\theta_{t+1}(F_{kt+1} + 1 - \delta) & \forall t \geq 0 \\
c_0: & V_{c0} & = & \theta_0 + \Phi \dfrac{\partial A_0}{\partial c_0} & \\
n_0: & V_{n0} & = & -\theta_0 F_{n0} + \Phi \dfrac{\partial A_0}{\partial n_0} &
\end{array}$$

- One can eliminate θ_t to obtain:

$$
\begin{aligned}
V_{ct} &= \beta V_{ct+1}(F_{kt+1} + 1 - \delta) \qquad \forall t \geq 1 \quad (6.9a)\\
V_{nt} &= -V_{ct} F_{nt} \qquad \forall t \geq 1 \quad (6.9b)\\
V_{c0} - \Phi \frac{\partial A_0}{\partial c_0} &= \beta V_{c1}(F_{k1} + 1 - \delta) \qquad (6.9c)\\
V_{n0} &= \underbrace{\left(\Phi \frac{\partial A_0}{\partial c_0} - V_{c0}\right)}_{\theta_0} F_{n0} + \Phi \frac{\partial A_0}{\partial n_0} \qquad (6.9d)
\end{aligned}
$$

plus the constraints

$$F(k_t, n_t) + (1-\delta)k_t = c_t + g_t + k_{t+1} \qquad (6.10a)$$

$$\sum_{t=0}^{\infty} \beta^t (U_{1t} c_t - U_2 n_t) = A_0 \qquad (6.10b)$$

$\rightsquigarrow$ solve for $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ and Φ

- Once allocations are obtained, we can find prices and taxes:

$$
\begin{aligned}
q_t^0 &= \beta^t \frac{U_{1t}}{U_{1,0}} && \rightsquigarrow q^0{}_t \\
r_t &= F_{kt} && \rightsquigarrow r_t \\
w_t &= F_{nt} && \rightsquigarrow w_t \\
(1-\tau_{nt}) w_t &= \frac{U_{2t}}{U_{1t}} && \rightsquigarrow \tau_{nt} \\
\frac{q_t^0}{q_{t+1}^0} &= (1-\tau_{kt+1}) r_{t+1} + 1 - \delta && \rightsquigarrow \tau_{kt+1}
\end{aligned}
$$

6.2. Zero capital tax at the steady state

- Assume $g_t = g \ \forall t \geq T$ and that there exists a steady state
- $V_{ct} \to \overline{V}_c$
- Then

$$V_{ct} = \beta V_{ct+1}(F_{kt+1} + 1 - \delta) \tag{6.9a}$$

  implies $\beta(F_k + 1 - \delta)) = 1$

- We also have $q_{0t} = \beta^t \frac{U_{1t}}{U_{1,0}} \rightsquigarrow \lim_{t\to\infty} q_t^0 / q_{t+1}^0 = 1/\beta$
- Then $\frac{q_t^0}{q_{t+1}^0} = (1 - \tau_{kt+1}) r_{t+1} + 1 - \delta \rightsquigarrow \beta((1 - \tau_k)F_k + 1 - \delta) = 1$
- This implies $\tau_k = 0$

7. Taxation of initial capital

- Up to now, we have assumed τ_{k0} given and small
- Assume now that τ_{k0} is a choice variable for the govt

$$\max \mathcal{J} = \sum_{t=0}^{\infty} \beta^t \Big(V(c_t, n_t, \Phi) + \theta_t \big(F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1} \big) \Big) - \Phi A_0$$

with

$$A_0(c_0, n_0, \tau_0^k, b_0) = U_{1,0} \Big(((1-\tau_{k0}) F_{k0} k_0 + 1 - \delta) k_0 + b_0 \Big) \tag{6.6}$$

- We have

$$\frac{\partial \mathcal{J}}{\partial \tau_{k0}} = \Phi U_{1,0} F_{k0} k_0$$

7. Taxation of initial capital

$$\frac{\partial \mathcal{J}}{\partial \tau_{k0}} = \Phi U_{1,0} F_{k0} k_0$$

- Φ = utility cost of raising resources through distortionary taxes
- As long as $\Phi > 0$, $\dfrac{\partial \mathcal{J}}{\partial \tau_{k0}} > 0$ and it is optimal to raise τ_{k0}
- $\tau_{k0} \nearrow \rightsquigarrow$ less distortive taxation $\rightsquigarrow \Phi \searrow$
- The govt will optimally increase τ_{k0} to the point where $\Phi = 0 =$ First best
- In the first best:
  - × $\tau_{kt} = \tau_{nt} = 0$
  - × The govt taxes $\tau_{k0} k_0$, lends it the the hh and pays g with the interest

