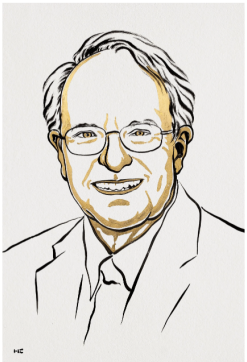
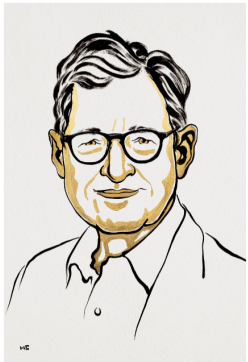
Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2026



Ill. Niklas Elmehed © Nobel Prize Outreach

Robert E. Hall

Prize share: 1/2



Ill. Niklas Elmehed © Nobel Prize Outreach

Michael Woodford

Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2026 was awarded to Robert E. Hall and Michael Woodford and "for microeconomic foundations of macroeconomic stabilisation".

Bob Hall

Hall, R. E. (1978). *Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence*. Journal of Political Economy 86(6), 971-987.

- Understanding consumption dynamics to understand stabilization policy
- Key contribution:
  - × if households' consumption is governed by their expectations of future income and other relevant considerations, then their decisions on consumption are the perfect indicator of their expectations:
  - × if they are optimistic, they will consume more than if they are pessimistic.
  - × This also means that they will only adjust their consumption if they change their assessment of the future, which requires the arrival of new and relevant information.
  - × Because, by definition, new information cannot be predicted – as then it would not be new – nor can changes in consumption be predicted, if the theory is correct.

Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, NJ: Princeton University Press.

- Key contribution: a New Keynesian macro model built upon microfoundations.
- 3-equation model:
  - × Role of intertemporal behaviour in consumption decision
  - × Microfoundation of a "new" Phillips curve.
  - × Role of monetary policy (TAYLOR rule)

2026-2027 – Econ 0107 – Macroeconomics I

Chapter 4 : Dynamic Optimization

(Chapter 4 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook)

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Changes from v1.0 in red
Changes from v1.1 in purple

Outline

1. Introduction
2. The Generic Problem
3. Sequential Methods: Finite Horizon
4. Sequential Methods: Infinite Horizon
5. Recursive Methods: Dynamic Programming
6. Dynamics in the NGM
7. Summary

1. Introduction

Motivation

Solow **model**: saving rate s is exogenous. **This chapter**: endogenize saving via optimization.

- Cass **(1965) and** Koopmans **(1963)**: Added microeconomic foundations to the Solow model $\Rightarrow$ the **optimizing neoclassical growth model (NGM)**.
  - $\times$ Same long-run implications as Solow, but explicit utility $\Rightarrow$ welfare analysis
  - $\times$ Forward-looking decisions $\Rightarrow$ richer policy implications
  - $\times$ With stochastic shocks: foundation of real business cycle theory (Chapter 14)
- Friedman **(1957)**: Individuals are forward-looking – decisions depend on expected future income, not just current income.
- Muth **(1961) and** Lucas **(1972)**: Expectations can be modelled as rational. Forecasts about uncertain future variables, those forecasts are consistent with the true underlying economic model, not arbitrary or systematically wrong.

Two solution methods:

1. **Sequential methods**: maximize over sequences
2. **Recursive methods** (dynamic programming): solve functional equations (useful for numerical analysis)

2. The Generic Problem

The Dynamic Optimization Problem

$$\max_{\{y_t, x_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \widehat{\mathcal{F}}(y_t) \quad \text{s.t.} \quad x_{t+1} = h(x_t, y_t), \quad x_{t+1} \in \Gamma(x_t), \quad x_0 \text{ given}$$

- y_t: **control variables** (consumption, hours, investment)
- x_t: **state variables** (capital, debt, housing stock)
- $\widehat{\mathcal{F}}$: per-period objective (utility or profit)
- β: discount factor; $1/\beta - 1$: discount rate [Stationarity: between t and $t+h$, discount depends only of h (β^h), not on t]
- h: law of motion linking states and controls
- $\Gamma(x_t)$: feasibility set
- x_0: given (exogenous initial condition)

Existence: Weierstrass theorem requires $\widehat{\mathcal{F}}$ continuous, Γ nonempty, closed, bounded.

Theorem 1 (Weierstrass Theorem)

If $f : S \to \mathbb{R}$ is continuous and $S \subset \mathbb{R}^n$ is nonempty and compact (closed and bounded), then f attains both its maximum and its minimum on S. More precisely, there exist $x^\star, x_\star \in S$ such that

$$f(x_\star) \;\leq\; f(x) \;\leq\; f(x^\star) \quad \text{for all } x \in S.$$

- Note that the supremum $\sup_{x \in S} f(x)$ and the infimum $\inf_{x \in S} f(x)$ always exist as extended real numbers, even without any of these conditions.
- What can fail is that these bounds are *attained*: there may be no point in S where f actually equals its supremum or infimum. The three conditions – continuity of f, closedness of S, and boundedness of S – are each individually necessary.

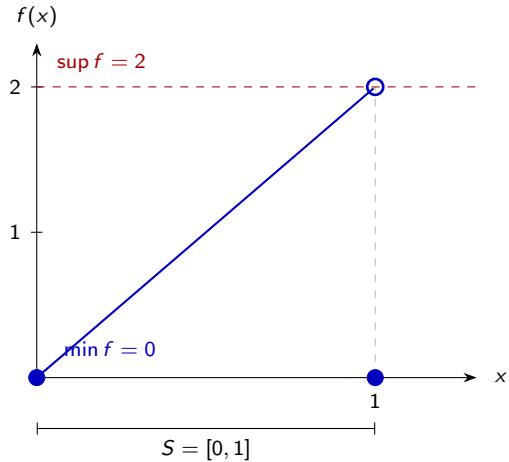
Examples in which Weierstrass Theorem fails

Discontinuous function on a compact set

Let $S = [0, 1]$ and define

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 1), \\ 0 & \text{if } x = 1. \end{cases}$$

- $\sup f = 2$ not attained (no maximum),
- $\min f = 0$ attained at $x = 0, 1$.



Examples in which Weierstrass Theorem fails

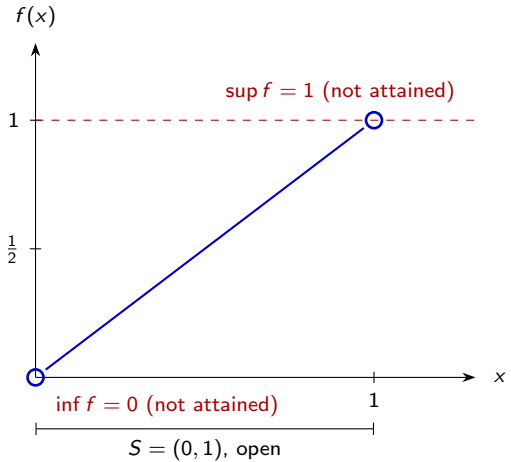
Continuous function on an open set

Let $S = (0,1)$ (open interval) and define $f(x) = x$.
f is continuous and the domain S is bounded. However, S is *not closed*: the boundary points 0 and 1 are excluded.

$$\sup_{x \in (0,1)} f(x) = 1 \quad \text{(not attained)},$$

$$\inf_{x \in (0,1)} f(x) = 0 \quad \text{(not attained)}.$$

Neither a maximum nor a minimum exists on $(0,1)$.



Examples in which Weierstrass Theorem fails

Continuous function on an unbounded set

Let $S = [0, \infty)$ and define $f(x) = \dfrac{x}{1+x}$.

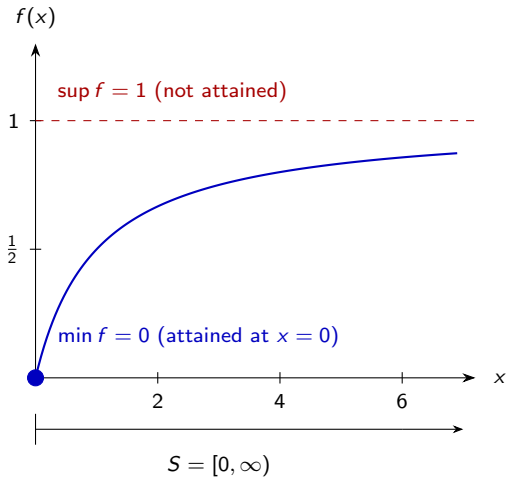
f is continuous, and S is closed and S is *not bounded*.

Minimum is attained: $\min f = 0$.

Supremum satisfies

$$\sup_{x \geq 0} f(x) = \lim_{x \to \infty} \frac{x}{1+x} = 1,$$

but the equation $f(x) = 1$ has no solution: $x/(1+x) = 1$ requires $x = \infty$. Hence f has no maximum on $[0, \infty)$.



Application 1: The Consumption-Saving Model

Agent lives T periods, maximizes $\sum_{t=0}^{T} \beta^t u(c_t)$.
Assumptions on u: strictly increasing, strictly concave, $\lim_{c \to 0} u'(c) = \infty$ (will make $c > 0$) , $u'(c) > 0$ (will make the budget constraint always binding).
Budget constraint: $c_t = w_t + (1 + r_t)a_t - a_{t+1}$
Borrowing limit: $a_{t+1} \geq \underline{a}, \quad a_{T+1} \geq 0$

Mapping to generic form:

- Control: c_t; State: a_t
- $a_{t+1} = h(a_t, c_t) = w_t + (1 + r_t)a_t - c_t$
- $\Gamma(a_t) = [\underline{a},\ w_t + (1 + r_t)a_t]$
- $\widehat{\mathcal{F}}(\cdot) = u(\cdot)$

Application 2: The Neoclassical Growth Model

Same utility. Production: $y_t = f(k_t)$. Capital accumulation:

$$k_{t+1} = (1-\delta)k_t + i_t$$

Resource constraint (closed economy, $i_t = y_t - c_t$):

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

Planner's problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.} \quad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t, \quad c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}$$

Mapping: Control: c_t; State: k_t; $\Gamma(k_t) = [0, f(k_t) + (1-\delta)k_t]$, $\widehat{\mathcal{F}}(\cdot) = u(\cdot)$.
$k_{t+1} = h(k_t, c_t) = f(k_t) + (1-\delta)k_t - c_t$.

Chapter 6: Under perfect competition, the competitive equilibrium solves this planning problem (welfare theorems).

3. Sequential Methods: Finite Horizon

Two-Period Consumption-Saving Model

- $T = 2$, $a_0 = 0$, $\underline{a} = 0$, constant r.
- Assume $w_0 > w_1$ and $\beta(1 + r) \geq 1 \rightsquigarrow$ interior solution ($a_1 > 0$).
- Budget constraints combine into the **lifetime budget constraint**:

$$c_0 + \frac{c_1}{1 + r} = w_0 + \frac{w_1}{1 + r}$$

Lagrangian is

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \mu\left\{w_0 + \frac{w_1}{1 + r} - c_0 - \frac{c_1}{1 + r}\right\} + \lambda\{w_0 - c_0\},$$

- The first-order conditions with respect to c_0 and c_1 are

$$\frac{\partial \mathcal{L}}{\partial c_0}: \quad u'(c_0) - \mu - \lambda = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial c_1}: \quad \beta u'(c_1) - \frac{\mu}{1 + r} = 0. \tag{2}$$

- The additional Kuhn-Tucker conditions are

$$\lambda[w_0 - c_0] = 0, \quad \underbrace{w_0 - c_0}_{a_1} \geq 0, \quad \text{and} \quad \lambda \geq 0. \tag{3}$$

- The first condition in (3): complementary slackness condition.
- Interior solution ($\lambda = 0$) yields the **Euler equation**:

$$u'(c_0) = \beta(1 + r)u'(c_1)$$

  - × LHS: marginal cost of saving one unit (less consumption today)
  - × RHS: discounted marginal benefit ($(1 + r)$ units tomorrow, valued at $\beta u'(c_1)$ each)
  - × When $\beta(1 + r) = 1$: full smoothing, $c_0 = c_1 = (w_0 + \beta w_1)/(1 + \beta)$

Generic T-Period Model: Result 1

If h is nonlinear, one cannot construct an intertemporal budget constraint (ibc), so we will use $x_{t+1} = h(x_t, y_t) \rightsquigarrow y_t = \widehat{h}(x_t, x_{t+1})$
Substitute the constraint into the objective: $F(x_t, x_{t+1}) \equiv \widehat{\mathcal{F}}(\widehat{h}(x_t, x_{t+1}))$.

$$\max_{\{x_{t+1} \in \Gamma(x_t)\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}).$$

Result 1: If F is jointly concave in (x_t, x_{t+1}), increasing in x_t, decreasing in x_{t+1}, and the interior conditions hold, then the optimum satisfies:
Euler equation:

$$F_2(x_t^\star, x_{t+1}^\star) + \beta F_1(x_{t+1}^\star, x_{t+2}^\star) = 0 \qquad \forall t < T$$

Terminal condition: $x_{T+1}^\star = \underline{x} = \min_{x_{T+1}} \Gamma(x_T)$

- $T + 2$ equations (Euler + initial + terminal), $T + 2$ unknowns
- Second-order difference equation in the state variable
- "Variational" condition: given x_t and x_{t+2}, vary x_{t+1} to achieve the best outcome

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \tag{P3}$$

$$\text{subject to} \quad k_{t+1} \geq 0 \quad \forall t \leq T, \quad k_0 \text{ given.}$$

The Lagrangian associated to our maximization problem (P3) is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \mu_t k_{t+1} \right\},$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}}: \quad -u'(c_t) + \beta u'(c_{t+1}) \left[f'(k_{t+1}) + 1 - \delta \right] + \mu_t = 0, \quad t \in \{0, \ldots, T-1\}$$

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}}: \quad -u'(c_T) + \mu_T = 0.$$

Kuhn-Tucker conditions also include $\mu_t k_{t+1} = 0$, $k_{t+1} \geq 0$, and $\mu_t \geq 0$, $0 \leq t \leq T$.

Finite-Horizon NGM: Euler Equation

FOC ($t = 0, \ldots, T-1$):

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta] + \mu_t$$

Terminal period: $u'(c_T) = \mu_T > 0 \Rightarrow k_{T+1} = 0$.
Interior Euler equation (Inada $\Rightarrow k_{t+1} > 0$ for $t < T$, so $\mu_t = 0$):

$$\underbrace{u'(c_t)}_{\substack{\text{marginal cost} \\ \text{of investment}}} = \underbrace{\beta\, u'(c_{t+1})}_{\substack{\text{utility increase} \\ \text{per unit return}}} \cdot \underbrace{\left[f'(k_{t+1}) + 1 - \delta\right]}_{\text{return on investment}}.$$

Solving by Backward Induction (Example 4.1)

Log utility, Cobb-Douglas $f(k) = Ak^\alpha$, full depreciation $\delta = 1$.
Euler equation for $t < T$ becomes

$$\frac{1}{Ak_t^\alpha - k_{t+1}} = \beta \frac{\alpha A k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}}.$$

Start from $k_{T+1} = 0$, use Euler equation backwards:

$$k_T = \frac{\alpha\beta}{1+\alpha\beta} Ak_{T-1}^\alpha, \qquad k_{T-1} = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta+(\alpha\beta)^2} Ak_{T-2}^\alpha, \quad \dots$$

General solution:

$$k_{t+1} = \alpha\beta \cdot \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} \cdot Ak_t^\alpha$$

Saving rate: $s_t = \alpha\beta \cdot \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}$ – depends on time (distance to T), but not on k_t. ($\lim_{T\to\infty} s_t = \alpha\beta$, constant as in SOLOW)

CRRA Utility and the EIS

Constant Relative Risk Aversion (CRRA) utility:

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \qquad \sigma > 0,\ \sigma \neq 1.$$

Special cases: $\sigma = 0$ (linear), $\sigma \to 1$ (log), $\sigma \to \infty$ (Leontief).
Elasticity of Intertemporal Substitution (EIS): How does c_{t+s}/c_t respond to a change in the return $R_{t,t+s}$?
With CRRA and linear production $f_t(k) = R_t k$:

$$\frac{c_{t+s}}{c_t} = (\beta^s)^{1/\sigma} (R_{t,t+s})^{1/\sigma} \quad \Rightarrow \quad \text{EIS} = \frac{\partial \log(c_{t+s}/c_t)}{\partial \log R_{t,t+s}} = \frac{1}{\sigma}$$

- $\sigma < 1$: substitution effect dominates; investment rises with returns
- $\sigma > 1$: income effect dominates; investment falls with returns
- $\sigma = 1$ (log): income and substitution effects exactly cancel

4. Sequential Methods: Infinite Horizon

Moving to Infinite Horizon

Advantage: Problem becomes **stationary** – the maximization at t is the same as at $t+1$ for the same capital level.

Dynastic interpretation: β^t = weight on descendants t generations ahead. $\beta < 1$: care more about self than offspring.

Mathematical complications:

1. **Unbounded utility**: Need $\beta < 1$, and with growth at rate γ, need $\beta(1+\gamma)^{1-\sigma} < 1$
2. **Constraint sets "too large"**: Need borrowing limits to prevent infinite debt accumulation
   - × $a_{t+1} \geq \underline{a}$ may be too restrictive (why would that be for *every* period).
   - × A terminal condition $a_{T+1} \geq \underline{a}$ is an alternative.

A Ponzi-Game

- Without a terminal period, need to rule out schemes where agent borrows and never repays.
- Assume. consumer is given initial asset $a_0 > 0$ and has no income. Period budget constraint ($r > 0$):

$$c_t + a_{t+1} = (1 + r)a_t$$

- Consider a candidate solution to the consumer's maximization problem $\{c_t^\star\}_{t=0}^\infty$. Absent further constraints, the agent could improve on $\{c_t^\star\}_{t=0}^\infty$ as follows:
  1. Let $\widetilde{c}_0 = c_0^\star + \varepsilon$, with $\varepsilon > 0$, making $\widetilde{a}_1 = a_1^\star - \varepsilon$.
  2. For every $t \geq 1$ leave $\widetilde{c}_t = c_t^\star$ by setting $\widetilde{a}_{t+1} = a_{t+1}^\star - \varepsilon(1 + r)^t$.
- Strictly increasing utility function $\rightsquigarrow$ the agent is clearly better off and satisfies the budget constraint period by period.
- This sort of improvement is possible for any candidate solution $\rightsquigarrow$ no maximum for lifetime utility.
- And the agent's debt is growing without bound at rate $1 + r$, and it is never repaid.
- This type of scheme, borrowing $(1 + r)a_t$ every period t to keep rolling the debt, is often called a "Ponzi scheme" or a "Ponzi game".

The No-Ponzi-Game Condition

- **No-Ponzi-game (nPg) condition**:

$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq 0$$

  Agent cannot have positive debt (in present value) at infinity.
- Take $c_0 + a_1 = (1+r)a_0$ and $c_1 + a_2 = (1+r)a_1$. This implies $c_0 + \frac{c_1}{1+r} = a_0 + \frac{a_2}{1+r}$.
- Substitute a_2 forward, etc...

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = a_0(1+r) - \frac{a_{T+1}}{(1+r)^T}.$$

- Taking limits, and using the nPg, we arrive at

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = a_0(1+r) - \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} \leq a_0(1+r),$$

- The agent cannot consume more (in NPV) that its wealth

Lifetime budget constraint

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq a_0(1+r)$$

- Ruling out Ponzi schemes is a natural infinite-horizon extension of the constraint $a_{T+1} \geq 0$ in a finite-horizon model.
- The "natural borrowing limit":
  - × Both in the finite-horizon and the infinite-horizon cases
  - × The loosest possible constraint on borrowing
  - × The only restriction is that you are able to pay back, in a present-value sense, if you set consumption to zero at all future times.
  - × In the particular model here, the natural borrowing limit is zero at all times
- The nPg condition and the lifetime budget constraint (and the natural borrowing limit, in this case $a_t \geq 0$ for all t) are equivalent.

The Transversality Condition

- **The transversality condition (TVC)**:

$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \leq 0$$

- **Interpretation**: It is never optimal to hold positive assets (in present value) at infinity – better to consume them.
- **nPg vs. TVC**:
  - × **nPg**: Externally imposed. Prevents over-*borrowing*.
  - × **TVC**: Self-imposed optimality condition.
- In the NGM, the TVC takes the form:

$$\lim_{t\to\infty} \beta^t u'(c_t)[f'(k_t) + 1 - \delta]k_t = 0$$

- In the NGM, if the economy converges to a steady state with $\beta < 1$: TVC satisfied automatically.

Sufficiency: Proposition 4.4

Proposition: If $F(x_t, x_{t+1})$ is jointly concave, Γ is convex, and $x_t^\star$ satisfies:
(i) Euler equation: $F_2(x_t^\star, x_{t+1}^\star) + \beta F_1(x_{t+1}^\star, x_{t+2}^\star) = 0$ for all t
(ii) TVC: $\lim_{t\to\infty} \beta^t F_1(x_t^\star, x_{t+1}^\star) x_t^\star = 0$
then $\{x_t^\star\}$ maximizes the objective.

Proof strategy (Appendix 4.A.3): Use the Euler equation repeatedly plus the definition of concavity ($f(x) \leq f(x^\star) + f'(x^\star)(x - x^\star)$) to show that any alternative feasible sequence gives a lower objective value – provided the TVC holds.

Solving the Infinite-Horizon NGM (Example 4.5)

Log utility, $f(k) = Ak^\alpha$, $\delta = 1$. Take the limit of the finite-horizon solution as $T \to \infty$:

$$k_{t+1} = \lim_{T\to\infty} \alpha\beta \cdot \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} \cdot Ak_t^\alpha = \alpha\beta Ak_t^\alpha$$

- Saving rate $s = \alpha\beta$ (constant – no longer depends on time!)
- Stationarity: infinite horizon $\Rightarrow$ same problem every period
- TVC: $\lim_{t\to\infty} \beta^t \alpha/(1-\alpha\beta) = 0$ since $\beta < 1$ ✓

Steady state: $\overline{k} = (\beta\alpha A)^{1/(1-\alpha)}$ – identical to SOLOW with $s = \alpha\beta$ and $\delta = 1$.

Guess and verify (Example 4.6): Guess $k_{t+1} = sAk_t^\alpha$, substitute into Euler equation $\Rightarrow s = \alpha\beta$.

Steady State (no growth)

- From the Euler equation with $c_t = c_{t+1}$:

$$1 = \beta(f'(\overline{k}) + 1 - \delta)$$

- Unique solution under Inada conditions. This is the **modified golden rule**: $f'(\overline{k}) = 1/\beta - 1 + \delta$.

Balanced Growth

- $Y_t = F(K_t, A_t L_t)$, A and L grow at rates γ and n
- CRRA preferences $u(c_t) = c_t^{1-\sigma}/(1-\sigma)$, $\sigma > 0$ and $\sigma \neq 1$.
- Utilitarian Social Planner: with $c = C/L$, objective is

$$\sum_{t=0}^{\infty} \beta^t L_t u(c_t),$$

- The resource constraint with growth is

$$C_t = F(K_t, A_t L_t) + (1-\delta)K_t - K_{t+1}.$$

- With "per efficiency units of labor" variables: $\widetilde{x}_t = X_t/(A_t L_t)$,

$$\widetilde{c}_t = f(\widetilde{k}_t) + (1-\delta)\widetilde{k}_t - (1+\gamma)(1+n)\widetilde{k}_{t+1}.$$

Balanced Growth

- Normalizing $A_0 = 1$ and $L_0 = 1$, we can rewrite the objective function as

$$\sum_{t=0}^{\infty} \beta^t L_t \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t (1+n)^t \frac{\left((1+\gamma)^t \widetilde{c}_t\right)^{1-\sigma}}{1-\sigma}$$
$$= \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{\widetilde{c}_t^{1-\sigma}}{1-\sigma},$$

  where now we use the adjusted discount factor $\widetilde{\beta}$:

$$\widetilde{\beta} \equiv \beta(1+n)(1+\gamma)^{1-\sigma}.$$

- Euler equation on BGP:

$$(1+n)(1+\gamma) = \widetilde{\beta}[f'(\overline{\widetilde{k}}) + 1 - \delta]$$

- **CRRA is the only utility consistent with balanced growth**.

5. Recursive Methods: Dynamic Programming

- **Key idea**: Break the infinite-dimensional problem into a sequence of identical small problems.
- Take advantage of stationnarity
- Take the infinite-horizon version of the problem (P1):

$$V(x_0) \equiv \max_{\{x_{t+1} \in \Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),$$

with x_0 given and break it into pieces:

$$
\begin{aligned}
V(x_0) &= \max_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \left\{ F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \beta \left[\max_{\{x_{t+1}\in\Gamma(x_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}) \right] \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \beta \left[\underbrace{\max_{\{x_{t+2}\in\Gamma(x_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^{t} F(x_{t+1}, x_{t+2})}_{\equiv V(x_1)} \right] \right\}
\end{aligned}
$$

The Bellman Equation

$$V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\}$$

- This is the **Bellman equation** – a functional equation (unknown is V, a function).
- Notation: $x_t \rightsquigarrow x$, $x_{t+1} \rightsquigarrow x'$.
- **Policy function**: $g(x) = \arg\max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\}$
  $\Rightarrow x' = g(x)$: stationary decision rule.

The NGM in Recursive Form

The standard NGM:

$$V(k) = \max_{k' \in [0, f(k)+(1-\delta)k]} \{u(f(k) + (1-\delta)k - k') + \beta V(k')\}$$

State: k (current capital). **Control**: k' (or equivalently c).

Other examples:

- **Periodic NGM** (alternating TFP A_h, A_l): Two value functions $V_h(k), V_l(k)$.
- **Delayed depreciation NGM** (capital depreciates after two periods): Two state variables (i_{-1}, i_{-2}).

Key lesson: Choosing state variables appropriately is essential for writing the problem recursively.

Properties of the Value Function

Under standard assumptions (F continuous, concave; Γ nonempty, compact, convex; $\beta < 1$):

1. **Contraction mapping**: Starting from any V_0, iterating $V_{n+1}(x) = \max_{x'}\{F(x, x') + \beta V_n(x')\}$ converges to V. Rate: $\|V_{n+1} - V\| \leq \beta \|V_n - V\|$.
2. V is **strictly increasing** (if F increasing in first argument and Γ monotone)
3. V is **strictly concave** (if F jointly strictly concave and Γ convex) $\Rightarrow$ policy g is unique
4. V is **differentiable** at interior solutions (if F is C^1)
5. g is **strictly increasing** (under all above conditions)

Property 1 is crucial for numerical computation (value function iteration, Chapter 10).

Solving the Bellman Equation (Example 4.7)

- Log utility, $f(k) = Ak^{\alpha}$, $\delta = 1$.
- Bellman equation is

$$V(k) = \max_{k' \geq 0} \{\log(Ak^{\alpha} - k') + \beta V(k')\}.$$

- Start from $V_0(k) = 0$:

$$V_1(k) = \max_{k'} \{\log(Ak^{\alpha} - k') + \beta V_0(k')\} = \log A + \alpha \log k \quad \text{(optimal: } k' = 0\text{)}$$

$$V_2(k) = \max_{k'} \{\log(Ak^{\alpha} - k') + \beta[\log A + \alpha \log k']\}$$

- FOC:

$$\frac{1}{Ak^{\alpha} - k'} = \frac{\beta\alpha}{k'} \implies k' = \frac{\alpha\beta Ak^{\alpha}}{1 + \alpha\beta}.$$

Solving the Bellman Equation (Example 4.7)

- Plug into the definition of V_2:

$$\begin{aligned} V_2(k) &= \log\left(Ak^\alpha - \frac{\alpha\beta Ak^\alpha}{1+\alpha\beta}\right) + \beta\left[\log A + \alpha \log \frac{\alpha\beta Ak^\alpha}{1+\alpha\beta}\right] \\ &= \underbrace{\log\left(A - \frac{\alpha\beta A}{1+\alpha\beta}\right) + \beta \log A + \alpha\beta \log \frac{\alpha\beta A}{1+\alpha\beta}}_{=a_2} + \underbrace{\left(\alpha + \alpha^2\beta\right)}_{=b_2} \log k. \end{aligned}$$

- Repeat and V_n will converge towards

$$V = \frac{1}{1-\beta} \cdot \frac{1}{1-\alpha\beta}\left[\log A + \log(1-\alpha\beta)^{1-\alpha\beta} + \log(\alpha\beta)^{\alpha\beta}\right] + \frac{\alpha}{1-\alpha\beta} \log k$$

Solving the Bellman Equation (Example 4.7)

- Alternative way: **Guess and verify**. Guess $V(k) = a + b \log k$.
- Bellman:

$$V(k) = \max_{k' \geq 0} \{ \log(Ak^\alpha - k') + \beta(a + b \log k') \}.$$

- FOC:

$$\frac{1}{Ak^\alpha - k'} = \frac{\beta b}{k'}$$

$\rightsquigarrow k' = \frac{\beta b A k^\alpha}{1+\beta b}$.

- Therefore

$$V(k) = \log\left(Ak^\alpha - \frac{\beta b A k^\alpha}{1+\beta b}\right) + \beta\left(a + b \log\left(\frac{\beta b A k^\alpha}{1+\beta b}\right)\right)$$

- Collect the constant terms and the terms in $\log k$:

$$V(k) = \underbrace{(1+\beta b)\log A - \log(1+\beta b) + \beta b \log \frac{\beta b}{1+\beta b} + \beta a}_{=a} + \underbrace{\alpha(1+\beta b)}_{=b} \log k$$

Solving the Bellman Equation (Example 4.7)

- The guess $V(k) = a + b \log k$ must hold **for all** $k \rightsquigarrow$ identify coefficients
- Coefficient on $\log k$:

$$b = \alpha(1 + \beta b) \implies b = \frac{\alpha}{1 - \alpha\beta}$$

$\rightsquigarrow 1 + \beta b = \frac{1}{1-\alpha\beta}, \quad \beta b = \frac{\alpha\beta}{1-\alpha\beta}, \quad \frac{\beta b}{1+\beta b} = \alpha\beta$

- Constant term:

$$a = \frac{1}{1 - \alpha\beta} \log A + \log(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \log(\alpha\beta) + \beta a$$

$$\implies a = \frac{1}{1 - \beta} \frac{1}{1 - \alpha\beta} \left[\log A + \log(1 - \alpha\beta)^{1-\alpha\beta} (\alpha\beta)^{\alpha\beta} \right]$$

- Policy: $k' = \frac{\beta b}{1+\beta b} A k^\alpha = \alpha\beta A k^\alpha$ – same answer as sequential methods.

The Functional Euler Equation and the Envelope Theorem: Intuition

- Bellman equation: $V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\}$.
- FOC wrt x':

$$F_2(x, x') + \beta V'(x') = 0. \qquad (\star)$$

- From the Bellman equation, on can compute $V'(x)$:

$$V'(x) = F_1(x, x') + \left(F_2(x, x') + \beta V'(x')\right) \frac{dx'}{dx} = 0.$$

- Because x' is also chosen optimally, $F_2(x, x') + \beta V'(x') = 0$ and

$$V'(x) = F_1(x, x') \qquad (\star\star)$$

- Therefore, using $(\star)$ and $(\star\star) \rightsquigarrow$ Functional Euler Equation (with $x' = g(x)$):

$$F_2(x, x') + \beta F_1(x', x'') = 0.$$

The Functional Euler Equation and the Envelope Theorem

- Bellman equation:
$$V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\}.$$
- Policy function $x' = g(x)$. Put in Bellman equation:
$$V(x) = F(x, g(x)) + \beta V(g(x)).$$
- Assuming an interior solution, x' satisfies the first-order condition
$$F_2(x, x') + \beta V'(x') = 0.$$
- Evaluated at the optimum, i.e., at $x' = g(x)$,
$$F_2(x, g(x)) + \beta V'(g(x)) = 0.$$
- In contrast to how we derived the sequential formulation, we now need to know the derivative of the value function $V'(\cdot)$ in order to solve for the policy rule.

The Functional Euler Equation and the Envelope Theorem

- Differentiate
$$V(x) = F(x, g(x)) + \beta V(g(x)).$$
- gives
$$V'(x) = F_1(x, g(x)) + g'(x) \underbrace{\left[F_2(x, g(x)) + \beta V'(g(x))\right]}_{\text{indirect effect through optimal choice of } x'}$$
- We have shown before that at the optimum
$$F_2(x, g(x)) + \beta V'(g(x)) = 0$$
- The indirect effect disappears (Envelope Theorem)
- We then have
$$V'(x) = F_1(x, g(x))$$

The Functional Euler Equation and the Envelope Theorem

$$\begin{aligned} F_2(x, g(x)) + \beta V'(g(x)) &= 0 && \text{(First-order condition)} \\ V'(x) &= F_1(x, g(x)) && \text{(Envelope condition)} \end{aligned}$$

- These two conditions imply $V'(g(x)) = F_1(g(x), g(g(x)))$
- **Functional Euler equation**:

$$F_2(x, g(x)) + \beta F_1(g(x), g(g(x))) = 0 \qquad \forall x$$

or

$$F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0, \quad \forall t,$$

- × Same as the sequential Euler equation, but written as a **functional equation** in g
- × Unknown is the function g (not a sequence)
- × Necessary condition only (may have multiple solutions); one needs to check the Bellman equation to select the correct one
- × Useful for numerical computation and for proving convergence

6. Dynamics in the NGM

Maximum Attainable Level of k

- There is a unique maximum attainable level of the capital stock, k_u.
- If the capital stock starts out below (or at) k_u, it cannot take a higher value than k_u in the future.
- Why? For any k, $f(k) + (1 - \delta)k$ is the highest feasible saving k' (zero consumption).
- k_u defined by $f(k_u) + (1 - \delta)k_u = k_u$ (discard solution $k = 0$)
- We can restrict attention to a search for value and policy functions over the closed and bounded set $[0, k_u]$.

- The standard NGM:

$$V(k) = \max_{k' \in [0, k_u]} \{u(f(k) + (1-\delta)k - k') + \beta V(k')\}$$

- With $g(k) = k'$, Euler equation is:

$$F_2(k, g(k)) + \beta F_1(g(k), g(g(k))) = 0 \qquad \forall k \in [0, k_u]$$

which is here

$$u'\big(f(k)+(1-\delta)k-g(k)\big) = \beta\, u'\big(f(g(k))+(1-\delta)g(k)-g(g(k))\big)\big(f'(g(k))+1-\delta\big).$$

- Note that steady state solves

$$1 = \beta\big(f'(\bar{k}) + 1 - \delta\big)$$

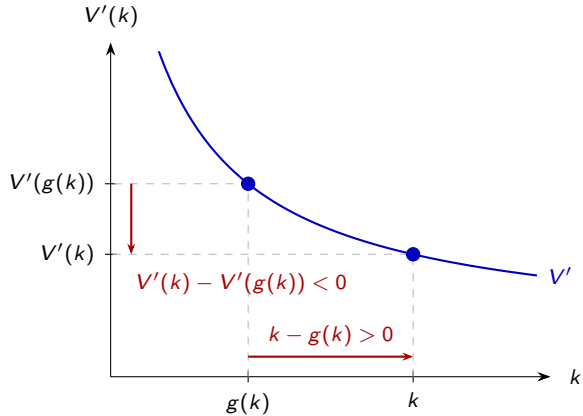
Proof of Global Convergence: Strict Concavity

- **Goal**: Show that $k_t \to \overline{k}$ from any $k_0 > 0$.
- Under the standard assumptions, $V(k)$ is strictly concave:

$$\left[V'(k) - V'(g(k))\right]\left[k - g(k)\right] \leq 0$$

$\forall k \in [0, k_u]$.

- V strictly concave $\Rightarrow$ V' strictly decreasing: if $k > g(k)$, then $V'(k) < V'(g(k))$, and conversely.
- The two brackets have opposite signs, and the product is 0 only if $g(k) = k$.



Proof of Global Convergence: Envelope Condition

$$\big[V'(k) - V'(g(k))\big]\big[k - g(k)\big] \leq 0$$

- Now let's find what are the values of $V'(k)$ and $V'(g(k)$
- Use the envelope theorem: $V'(k) = F_1(k, g(k))$, with $F(k, k') = u\big(f(k) + (1-\delta)k - k'\big)$.
- which gives

$$V'(k) = u'\big(f(k) + (1-\delta)k - g(k)\big)\big(f'(k) + 1 - \delta\big).$$

Proof of Global Convergence: the First-Order Condition

- Bellman equation of the growth model, with $c = f(k) + (1-\delta)k - k'$:

$$V(k) = \max_{k' \in [0, k_u]} \Big\{ u\big(f(k) + (1-\delta)k - k'\big) + \beta V(k') \Big\},$$

  i.e. $F(k, k') = u\big(f(k) + (1-\delta)k - k'\big)$, so $F_2(k, k') = -u'(c)$.
- First-order condition $F_2(k, k') + \beta V'(k') = 0$:

$$-u'(c) + \beta V'(k') = 0.$$

- Evaluated at the optimum $k' = g(k)$ and solved for V':

$$V'(g(k)) = \frac{1}{\beta}\, u'\big(f(k) + (1-\delta)k - g(k)\big).$$

- Both $V'(k)$ and $V'(g(k))$ are now proportional to the same $u'(c) \Rightarrow$ we can factorize $u'(c)$ in the concavity inequality.

Proof of Global Convergence

- Inserting these two expressions into the left-most bracket of the inequality and factorizing u', we obtain

$$\underbrace{u'\big(f(k)+(1-\delta)k-g(k)\big)}_{\text{positive}}\left[f'(k)+1-\delta-\frac{1}{\beta}\right]\big[k-g(k)\big]\leq 0 \quad \forall k\in[0,k_u]. \tag{4}$$

  - × $k=\overline{k}$: $f'(\overline{k})+1-\delta=1/\beta$ and $g(\overline{k})=\overline{k}$ ✓
  - × $k>\overline{k}$: $f'(k)+1-\delta<1/\beta \Rightarrow g(k)<k$ (capital falls)
  - × $k<\overline{k}$: $f'(k)+1-\delta>1/\beta \Rightarrow g(k)>k$ (capital rises)
- We have global monotonic convergence

Local Approximation of the Policy Function

- Approximate $g(k) \approx a_0 + a_1 k$ near $\overline{k}$.
- From $g(\overline{k}) = \overline{k}$: $a_0 = \overline{k}(1 - a_1) \rightsquigarrow$ one restriction for a_0 and a_1
- Functional Euler equation:

$$u'\big(f(k)+(1-\delta)k-g(k)\big) = \beta\, u'\big(f(g(k))+(1-\delta)g(k)-g(g(k))\big)\big(f'(g(k))+1-\delta\big).$$

- Differentiate the functional Euler equation w.r.t. k, evaluate at $\overline{k}$:

$$\mathrm{LHS}_k(\bar{k}) = u''(\bar{c})\left[\underbrace{f'(\bar{k}) + 1 - \delta}_{1/\beta} - \underbrace{g'(\bar{k})}_{=a_1}\right]$$

$$\mathrm{RHS}_k(\bar{k}) = \beta u''(\bar{c})\left[(f'(\bar{k}) + 1 - \delta)a_1 - \underbrace{g'(\bar{k})g'(\bar{k})}_{=a_1^2}\right]\left[f'(\bar{k}) + 1 - \delta\right] + \beta u'(\bar{c}) f''(\bar{k}) a_1.$$

- We then have

$$u''(\overline{c})[1 - \beta a_1] = u''(\overline{c})[a_1 - \beta a_1^2] + \beta^2 u'(\overline{c}) f''(\overline{k}) a_1$$

Local Approximation of the Policy Function

$$u''(\overline{c})[1-\beta a_1] = u''(\overline{c})[a_1 - \beta a_1^2] + \beta^2 u'(\overline{c}) f''(\overline{k}) a_1$$

- This is a quadratic equation in a_1
- Define $\Theta \equiv \beta^2 u'(\overline{c}) f''(\overline{k}) / u''(\overline{c}) > 0$. Solving the quadratic:

$$a_1 = \frac{1+\beta+\Theta - \sqrt{(1+\beta+\Theta)^2 - 4\beta}}{2\beta}$$

and $a_0 = \overline{k}(1-a_1)$

- With CRRA, $u(c) = (c^{1-\sigma}-1)/(1-\sigma)$:

$$\Theta = -\frac{\beta^2 f''(\bar{k})\big(f(\bar{k}) - \delta\bar{k}\big)}{\sigma}.$$

- × $a_1 \in (0,1]$: the correct root (monotone convergence)
- × a_1 decreasing in Θ (faster convergence when Θ is large)
- × σ large (unwilling to change c) $\Rightarrow \Theta$ small $\Rightarrow a_1$ close to 1 $\Rightarrow$ slow convergence
- × $\sigma = 0$ (linear utility) $\Rightarrow \Theta \to \infty \Rightarrow a_1 \to 0 \Rightarrow$ immediate convergence

Calibration: The Role of σ

With CRRA $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ and Cobb-Douglas $f(k) = k^{\alpha}$:

$$\Theta = \frac{\beta^2(1-\alpha)}{\sigma}\left(\frac{1}{\beta} - 1 + \delta\right)\left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right) - \delta\right)$$

Θ is positive, decreasing in σ, decreasing in α, increasing in δ.

Empirical estimates of $1/\sigma$ (EIS):

- Hall (1988), aggregate data: $1/\sigma \approx 0.1$
- Attanasio and Weber (1993, 1995), micro data: $1/\sigma \in [0.3, 0.8]$
- Most applied macro: $1/\sigma = 1$ (log) or $1/\sigma = .5$

Result: Convergence in the NGM is qualitatively like Solow but quantitatively different – the endogenous saving rate depends on k through σ.

7. Summary

Key Takeaways (I)

1. **Generic problem**: Controls y_t, states x_t, discount factor β, feasibility Γ
2. **Euler equation**: $F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$ – marginal cost of saving = marginal benefit
3. **Terminal condition** (finite horizon): $x_{T+1} = \underline{x}$ (consume everything)
4. **nPg condition** (infinite horizon): $\lim a_{t+1}/(1+r)^t \geq 0$ (no exploding debt)
5. **TVC**: $\lim \beta^t F_1 x_t = 0$ (no over-accumulation); self-imposed optimality condition

Key Takeaways (II)

6. **CRRA utility**: EIS $= 1/\sigma$; the only utility consistent with balanced growth
7. **Bellman equation**: $V(x) = \max_{x'}\{F(x, x') + \beta V(x')\}$; contraction mapping $\Rightarrow$ unique V, can be found by iteration
8. **Properties of** V: increasing, strictly concave, differentiable at interior; policy g unique and increasing
9. **Functional Euler equation**: $F_2(x, g(x)) + \beta F_1(g(x), g(g(x))) = 0$; necessary condition from Envelope Theorem
10. **Global convergence**: Proved using strict concavity of V + Envelope Theorem; speed depends on σ (curvature of utility) and α (curvature of production)
11. **Local approximation**: $g(k) \approx a_0 + a_1 k$; a_1 solved from a quadratic; $a_1 \in (0, 1)$ for the stable root

