2026-2027 – Econ 0107 – Macroeconomics I

Chapter 5 : Dynamic Competitive Equilibrium

(Chapter 5 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook)

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1. Introduction

What Is a Competitive Equilibrium?

- Two conditions must hold simultaneously:
  1. Agents maximize their objectives subject to constraints, taking prices as given
  2. Markets clear: demand equals supply for all traded goods
- **Three formulations**:
  1. Arrow-Debreu: all trades at date 0; prices $\{p_t\}_{t=0}^{\infty}$
  2. **Sequential**: trades period by period; bond price q_t each period
  3. **Recursive**: prices and policies as functions of state variables
- For many economies, all three yield the **same allocations**. The first two express equilibria as sequences; the third as functions.

2. Arrow-Debreu Equilibrium

Endowment Economy: Primitives

- Continuum of consumers $i \in [0, 1]$; each is endowed with $y_{i,t} \in \mathbb{R}_+$ of a single non-storable good
- Common preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_{i,t}),$$

  with u strictly increasing and strictly concave
- Price p_t: price of time-t good in terms of time-0 good; normalize $p_0 = 1$
- Budget constraint (equality since u strictly increasing):

$$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t y_{i,t}$$

- Contracts are fully enforcable

Definition 1

Definition 1

An ARROW-DEBREU *competitive equilibrium is a set of sequences* $\{c^\star_{i,t}\}_{t=0}^\infty$*, for each* $i \in [0,1]$*, and* $\{p_t\}_{t=0}^\infty$ *such that*

1. *for each* i, $\{c^\star_{i,t}\}_{t=0}^\infty$ *solves*

$$\max_{\{c_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t) \quad s.t. \quad \sum_{t=0}^\infty p_t c_t = \sum_{t=0}^\infty p_t y_{i,t}$$

2. *for each* t,

$$\int_0^1 c^\star_{i,t}\, di = \int_0^1 y_{i,t}\, di$$

- Point 1: optimization given prices. Point 2: choosing prices such that goods market clears.

Characterization: Prices

- FOC from the Lagrangian on the lifetime budget:

$$\beta^t u'(c_{i,t}) = \lambda_i p_t$$

  where λ_i is the multiplier on the lifetime budget constraint. It does not depend on t.

- Eliminating λ_i between $t = 0$ and a generic t:

$$p_t = \beta^t \frac{u'(c_{i,t})}{u'(c_{i,0})}$$

- Since p_t is common to all agents: the ratio of marginal utilities at different dates is **equalized across all consumers**.

Log Utility: Prices and Consumption Growth

- With $u(\cdot) = \log(\cdot)$: $p_t = \beta^t c_{i,0}/c_{i,t}$. Multiply by $c_{i,t}$, integrate over i, use market clearing ($\int_0^1 c_{i,t}\, di = Y_t$):

$$p_t\, c_{i,t} = \beta^t c_{i,0} \;\Rightarrow\; p_t \int_0^1 c_{i,t}\, di = \beta^t \int_0^1 c_{i,0}\, di \;\Rightarrow\; p_t Y_t = \beta^t Y_0$$

$$p_t = \beta^t \frac{Y_0}{Y_t}$$

  p_t is low if t is large or if the endowment is large.

- All consumers share the same consumption growth:

$$\frac{c_{i,t+1}}{c_{i,t}} = \frac{Y_{t+1}}{Y_t} = \beta \frac{p_t}{p_{t+1}} \quad \text{for all } i$$

- If $Y_t = \bar{Y}$ constant: $c_{i,t}$ is constant for all i and t (**perfect consumption smoothing**).

Consumption Levels

- From the Euler equations and the budget constraint:

$$\underbrace{\sum_{t=0}^{\infty} p_t c_{i,t}}_{=\sum_t \beta^t c_{i,0} = \frac{c_{i,0}}{1-\beta}} = \underbrace{\sum_{t=0}^{\infty} p_t y_{i,t}}_{=Y_0 \sum_t \beta^t \frac{y_{i,t}}{Y_t}} \quad \text{using } p_t c_{i,t} = \beta^t c_{i,0} \text{ and } p_t = \beta^t \frac{Y_0}{Y_t}$$

- Hence:

$$c_{i,0} = (1-\beta) Y_0 \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t}$$

Consumption Shares

- Individual i's share of aggregate consumption is constant over time:

$$c_{i,t} = \beta^t \frac{c_{i,0}}{p_t} = \beta^t c_{i,0} \frac{Y_t}{\beta^t Y_0} = \frac{c_{i,0}}{Y_0} Y_t$$

$$c_{i,t} = \theta_i Y_t, \qquad \theta_i = (1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t}$$

- Endowments in periods with high Y_t receive lower weight (marginal utility is lower).
- Two consumers with the same average endowments can have different wealth: the wealthy consumer is the one whose endowments are high when others' endowments are low.
- **General CRRA**: $p_t = \beta^t (Y_0/Y_t)^\sigma$. Higher $\sigma \Rightarrow$ resources more valuable in low-endowment periods.

Definition 2

Definition 2 (Representative Agent)

When all agents have equal endowments and u is strictly concave, all consumption choices are identical:
An AD CE is $\{c_t^\star\}_{t=0}^\infty$ and $\{p_t\}_{t=0}^\infty$ such that

1. $\{c_t^\star\}_{t=0}^\infty$ *solves*

$$\max_{\{c_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t) \quad s.t. \quad \sum_{t=0}^\infty p_t c_t = \sum_{t=0}^\infty p_t y_t$$

2. $c_t^\star = y_t$

Production Economy with Labor: Primitives

- Household i: efficiency units e_i, hours $\ell_{i,t}$, effective hours $e_i \ell_{i,t}$, utility:

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_{i,t}) - v(\ell_{i,t}) \right],$$

  with v increasing and convex
- Firm: $Y_t = A_t L_t$ (CRS), hires labor at wage w_t per efficiency unit
- Firm profit max:

$$\max_{\{L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \{ p_t A_t L_t - p_t w_t L_t \}$$

- Household budget:

$$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t w_t e_i \ell_{i,t}$$

Definition 3

Definition 3

An AD CE is $\{c^\star_{i,t}\}_{t=0}^\infty$, $\{\ell^\star_{i,t}\}_{t=0}^\infty$ *for each* $i \in [0,1]$, $\{L^\star_t\}_{t=0}^\infty$, $\{p_t\}_{t=0}^\infty$, $\{w_t\}_{t=0}^\infty$ *such that*

1. *for each* i, $\{c^\star_{i,t}\}_{t=0}^\infty$ *and* $\{\ell^\star_{i,t}\}_{t=0}^\infty$ *solve*

$$\max_{\{c_t,\ell_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t[u(c_t) - v(\ell_t)] \quad s.t. \quad \sum_{t=0}^\infty p_t c_t = \sum_{t=0}^\infty p_t w_t e_i \ell_t$$

2. $\{L^\star_t\}_{t=0}^\infty$ *solves*

$$\max_{\{L_t\}_{t=0}^\infty} \sum_{t=0}^\infty \{p_t A_t L_t - p_t w_t L_t\}$$

3. *for each* t, *good and labour market clear:*

$$\int_0^1 e_i \ell^\star_{i,t}\, di = L^\star_t \qquad \text{and} \qquad \int_0^1 c^\star_{i,t}\, di = A_t L^\star_t$$

- **Household FOCs**:

$$\beta^t u'(c_{i,t}) = \lambda_i p_t, \qquad v'(\ell_{i,t}) = e_i w_t u'(c_{i,t})$$

- **Firm Optimal Behavior**: For an interior solution with CRS, $w_t = A_t$. If $A_t > w_t$: no finite optimum. If $A_t < w_t$: firm shuts down.
- With log utility and $w_t = A_t$: the intratemporal FOC becomes $v'(\ell_{i,t})c_{i,t} = e_i A_t$. With $c_{i,t} = \theta_i Y_t$ and conjecturing $\theta_i = e_i$: all agents supply the same labor $\ell_{i,t} = L_t$. Income and substitution effects exactly cancel.

With heterogeneous initial assets

- Individual i is endowed with $a_{i,0}$ at time zero

$$\int_0^1 a_{i,0}\, di = 0$$

- Agents with different asset holdings will work different amounts because assets generate an income effect but not a substitution effect
- Consider the simple static case
  - × Budget constraint

$$c_i = e_i A \ell_i + a_i$$

  - × First-order condition, be given by

$$v'(\ell_i) = e_i A / (e_i A \ell_i + a_i)$$

  - × Individual labor supply depends nonlinearly on a_i
  - × Total labor supply and total consumption, will depend on the distribution of assets (inequalities shape macro allocations)

Neoclassical Growth Economy: Primitives

- Production:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- Capital:

$$k_{i,t+1} = (1-\delta)k_{i,t} + \iota_{i,t}$$

- Firm:

$$\max_{\{K_t, L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \{p_t A_t K_t^{\alpha} L_t^{1-\alpha} - p_t r_t K_t - p_t w_t L_t\}$$

- Household budget:

$$\sum_{t=0}^{\infty} p_t(c_{i,t} + \iota_{i,t}) = \sum_{t=0}^{\infty} p_t(r_t k_{i,t} + w_t e_i \ell_{i,t})$$

Definition 4

Definition 4

An AD CE is $\{c_{i,t}^\star\}$, $\{\iota_{i,t}^\star\}$, $\{\ell_{i,t}^\star\}$ *for each* i, $\{L_t^\star\}$, $\{K_{t+1}^\star\}$, $\{p_t\}$, $\{w_t\}$, $\{r_t\}$ *such that*

1. *for each* i, $\{c_{i,t}^\star\}$, $\{\iota_{i,t}^\star\}$, $\{\ell_{i,t}^\star\}$ *solve*

$$\max_{\{c_t,\iota_t,\ell_t\}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(\ell_t)]$$

$$s.t. \quad \sum_{t=0}^{\infty} p_t(c_t + \iota_t) = \sum_{t=0}^{\infty} p_t(r_t k_t + w_t e_i \ell_t)$$

   where $k_{t+1} = (1-\delta)k_t + \iota_t$, $k_0 = k_{i,0}$

2. $\{L_t^\star\}$, $\{K_t^\star\}$ *(with* $K_0^\star = \int_0^1 k_{i,0}\, di$*) solve*

$$\max_{\{K_t,L_t\}} \sum_{t=0}^{\infty} \{p_t A_t K_t^\alpha L_t^{1-\alpha} - p_t r_t K_t - p_t w_t L_t\}$$

3. $\int_0^1 e_i \ell_{i,t}^\star\, di = L_t^\star$, $\int_0^1 k_{i,t}^\star\, di = K_t^\star$, $\int_0^1 (c_{i,t}^\star + \iota_{i,t}^\star)\, di = A_t (K_t^\star)^\alpha (L_t^\star)^{1-\alpha}$

Optimality Conditions for Definition 4

- **Consumption and labor** (same structure as before):

$$\beta^t u'(c_{i,t}) = \lambda_i p_t, \qquad v'(\ell_{i,t}) = e_i w_t u'(c_{i,t})$$

- **Investment** (new): the household's FOC for $\iota_{i,t}$ yields

$$p_t = p_{t+1} r_{t+1} + (1-\delta) p_{t+1} = p_{t+1}[r_{t+1} + 1 - \delta]$$

  Cost of one unit of capital today = rental revenue + undepreciated value tomorrow.

- **Firm FOCs**: $r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$, $w_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}$. CRS $\Rightarrow$ zero profits.

Equivalence with the Planner

- With identical agents ($e_i = 1$, $k_{i,0} = K_0$) and log utility, the aggregate equilibrium conditions become:

$$\frac{C_{t+1}}{C_t} = \beta[1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha}]$$

$$v'(L_t) = \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}}{C_t}$$

$$C_t + K_{t+1} = A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t$$

- These are identical to the FOCs of the social planner from Chapter 4 $\Rightarrow$ **First Welfare Theorem**.

3. Sequential Equilibrium

- Trades occur period by period; bond price q_t (pay q_t today, receive 1 tomorrow)
- Gross interest rate:
$$1 + r_{t+1} = 1/q_t$$
- Period budget:
$$c_{i,t} + q_t a_{i,t+1} = y_{i,t} + a_{i,t}$$
- Need a no-Ponzi-game condition to rule out infinite borrowing
- Initial assets:
$$\int_0^1 a_{i,0}\, di = 0$$

Definition 5 (Sequential CE, Endowment Economy)

A sequential CE is $\{c^\star_{i,t}\}_{t=0}^{\infty}$, $\{a^\star_{i,t+1}\}_{t=0}^{\infty}$ *for each* $i \in [0,1]$, *and* $\{q_t\}_{t=0}^{\infty}$ *such that*

1. *for each* i, $\{c^\star_{i,t}\}_{t=0}^{\infty}$ *and* $\{a^\star_{i,t+1}\}_{t=0}^{\infty}$ *solve*

$$\max_{\{c_{i,t},a_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

$$s.t. \quad c_{i,t} + q_t a_{i,t+1} = a_{i,t} + y_{i,t} \ \text{ for all } t,$$

$$\lim_{t\to\infty} \left(\prod_{s=0}^{t} q_s \right) a_{i,t+1} \geq 0 \quad (nPg)$$

2. *for all* t,

$$\int_0^1 c^\star_{i,t}\, di = \int_0^1 y_{i,t}\, di \qquad \text{and} \qquad \int_0^1 a^\star_{i,t+1}\, di = 0$$

- The second clearing condition says aggregate assets net out to zero. By Walras' Law, only one clearing condition is needed.

Sequential FOCs

- FOCs from the Lagrangian with period-by-period budget constraints:

$$\beta^t u'(c_{i,t}) = \lambda_{i,t}, \qquad q_t \lambda_{i,t} = \lambda_{i,t+1}$$

- Unlike the AD case, $\lambda_{i,t}$ now depends on t (one multiplier per period budget).
- Eliminating multipliers:

$$q_t = \beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})}$$

Equivalence: Sequential $\Leftrightarrow$ Arrow-Debreu

- Chaining from 0 to $t-1$:

$$\prod_{s=0}^{t-1} q_s = \prod_{s=0}^{t-1} \beta \frac{u'(c_{i,s+1})}{u'(c_{i,s})} = \beta^t \, \frac{u'(c_{i,1})}{u'(c_{i,0})} \cdot \frac{u'(c_{i,2})}{u'(c_{i,1})} \cdots \frac{u'(c_{i,t})}{u'(c_{i,t-1})}$$

$$\prod_{s=0}^{t-1} q_s = \beta^t \frac{u'(c_{i,t})}{u'(c_{i,0})} = p_t$$

  The product of sequential bond prices equals the Arrow-Debreu price. Also: $p_{t-1}/p_t = 1/q_{t-1} = 1 + r_t$.

- Consolidating period budgets forward and using TVC (so the asset term vanishes):

$$\sum_{t=0}^{\infty} p_t c_{i,t} = a_{i,0} + \sum_{t=0}^{\infty} p_t y_{i,t}$$

- Same lifetime budget $\Rightarrow$ same allocations.

Sequential NGM, Inelastic Labour Supply

Definition 6 (Sequential CE, NGM, Inelastic Labour Supply)

A sequential CE is $\{c_{i,t}^\star\}_{t=0}^\infty$, $\{k_{i,t+1}^\star\}_{t=0}^\infty$ *for each* $i \in [0,1]$, $\{K_t^\star\}$, $\{L_t^\star\}$ $\{r_t\}_{t=0}^\infty$, $\{w_t\}_{t=0}^\infty$ *such that*

1. *for each* i, $\{c_{i,t}^\star\}_{t=0}^\infty$ *and* $\{k_{i,t+1}^\star\}_{t=0}^\infty$ *solve*

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

$$s.t. \quad c_{i,t} + k_{i,t+1} = (1 - \delta + r_t)k_{i,t} + w_t e_i \text{ for all } t,$$

$$\lim_{t\to\infty} \frac{k_{i,t+1}}{\prod_{s=0}^{t}(1-\delta+r_{s+1})} \geq 0 \quad (nPg)$$

2. *for each* t, $(K_t^\star, L_t^\star)$ *solve* $\max_{K,L} A_t K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t$
3. *for each* t, $C_t + K_{t+1} = A_t K_t^\alpha + (1-\delta)K_t$, *where* $C_t = \int_0^1 c_{i,t}^\star \, di$; $K_t^\star = \int_0^1 k_{i,t}^\star \, di$, $L_t^\star = 1$

- By Walras' Law, $C_t + K_{t+1} = A_t K_t^\alpha + (1-\delta)K_t$ is redundant given condition 2 and the budget constraints of condition 1.

4. Recursive Equilibrium

Recursive Methods: Overview

- Express equilibrium as **functions** of state variables, not sequences indexed by time.
- A state variable must be both **relevant** (determines prices) and **predetermined**.
- Two steps:
  1. **Steady-state** recursive equilibria: aggregate variables constant; functions specify that agents *choose* to remain at constant levels
  2. **Dynamic** recursive equilibria: aggregates change with the state K; agents use K to forecast future prices via $r(K)$, $w(K)$, $K' = G(K)$

Steady-State CE, Endowment Economy

Definition 7 (Steady-State CE, Endowment Economy)

With constant endowments y_i for each i, a recursive steady-state CE is a q and a set of functions $V_i^\star(a)$ and $g_i^\star(a)$, and asset values $a_i^\star$, for each $i \in [0,1]$, such that

1. *for each i, $V_i^\star(a)$ solves*

$$V_i(a_i) = \max_{a_i'} \{u(a_i + y_i - qa_i') + \beta V_i(a_i')\} \quad \forall a_i$$

   with $g_i^\star(a_i)$ attaining the max for all a_i

2. *for each i, $g_i^\star(a_i^\star) = a_i^\star$ (steady state)*

3. $\int_0^1 g_i^\star(a_i^\star)\, di = 0$ *(asset market clearing)*

Characterization of Steady-State CE, Endowment Economy

$$V_i(a_i) = \max_{a_i'} \left\{ u(a_i + y_i - qa_i') + \beta V_i(a_i') \right\} \quad \forall a$$

- FOC wrt a_i':

$$qu'(a_i + y_i - qa_i') = \beta V_i'(a_i')$$

- Envelope condition

$$V_i'(a_i) = u'(a_i + y_i - qa_i')$$

  which implies

$$V_i'(a_i') = u'(a_i' + y_i - qa_i'')$$

- now use

$$a_i' = g_i(a_i)$$

Characterization of Steady-State CE, Endowment Economy

- The functional Euler equation (FOC + Envelope Theorem), for all i:

$$qu'(a_i + y_i - qg_i(a_i)) = \beta u'(g_i(a_i) + y_i - qg_i(g_i(a_i)))$$

- Evaluating at $a_i = a_i^\star$ with stationarity ($g_i^\star(a_i^\star) = a_i^\star$):

$$q = \beta$$

- **Policy**: $g_i^\star(a_i^\star) = a_i^\star$ for all i (permanent-income result in recursive form).
- **Value**: $V_i^\star(a_i^\star) = u(a_i^\star(1-q) + y_i)/(1-\beta)$.
- The asset distribution $\{a_i^\star\}$ is indeterminate subject to $\int_0^1 a_i^\star \, di = 0$. This model has no predictions for long-run wealth distributions (Chapter 21 addresses this).

Steady-State CE, NGM, Inelastic Labour Supply

Definition 8 (Steady-State CE, NGM, Inelastic Labour Supply)

A recursive steady-state CE consists of scalars r and w and a set of functions $V_i^\star(k_i)$ and $g_i^\star(k_i)$, and capital holdings $k_i^\star$, for each $i \in [0,1]$, such that

1. *for each i, $V_i^\star(k_i)$ solves*

$$V_i(k_i) = \max_{k_i'} \{u((1-\delta+r)k_i + we_i - k_i') + \beta V_i(k_i')\} \quad \forall k_i$$

   with $g_i^\star(k_i)$ attaining the max for all k_i

2. *$(K^\star, L^\star)$ solves*

$$\max_{K,L} F(K,L) - rK - wL$$

3. *for each i, $g_i^\star(k_i^\star) = k_i^\star$ (steady state)*

4. *$\int_0^1 k_i^\star \, di = K^\star$ and $L^\star = 1$*

Characterization of Steady-State CE, NGM, Inelastic Labour Supply

- From the Euler equation at stationarity:

$$\beta = 1/(1 - \delta + r),$$

  which pins down r.
- Then $r = F_1(K^\star, 1)$ determines $K^\star$, and $w = F_2(K^\star, 1)$ determines w.
- **Policies**: $g_i^\star(k_i^\star) = k_i^\star$ and

$$V_i^\star(k_i^\star) = u(k_i^\star(r - \delta) + we_i)/(1 - \beta)$$

- The distribution of capital is indeterminate, subject to

$$\int_0^1 k_i^\star \, di = K^\star.$$

- Assume A_t constant and that all agents have same k_{i0} and $e_i \rightsquigarrow$ they will all behave the same $\rightsquigarrow$ distributions are degenerate, with all mass concentrated at the mean.
- Out of steady state: prices depend on aggregate capital K (relevant and predetermined).
- Representative agent takes as given: price functions $r(K)$, $w(K)$, and a law of motion $K' = G(K)$.
- Individual Bellman equation (must allow $k_i \neq K$):

$$V(k_i, K) = \max_{k_i'} \left\{ u((1 - \delta + r(K))k_i + w(K) - k_i') + \beta V(k_i', G(K)) \right\}$$

for all (k_i, K). Agent tracks K because prices depend on it.

Recursive CE, NGM, Inelastic Labour Supply

Definition 9 (Recursive CE, NGM, Inelastic Labour Supply)

A recursive CE consists of functions $r(K)$, $w(K)$, $G^\star(K)$, $V^\star(k_i, K)$, and $g^\star(k_i, K)$ such that

1. $V^\star(k_i, K)$ *solves*

$$V(k_i, K) = \max_{k_i'} \{u((1 - \delta + r(K))k_i + w(K) - k_i') + \beta V(k_i', G^\star(K))\}$$

   for all (k_i, K), with $k_i' = g^\star(k_i, K)$ attaining the max

2. $r(K) = F_1(K, 1)$ *and* $w(K) = F_2(K, 1)$ *for all* K

3. $G^\star(K) = g^\star(K, K)$ *for all* K *(consistency)*

- Consistency: aggregate capital evolves according to individual choices when each agent holds $k_i = K$. The resource constraint is automatically satisfied.

Functional Euler Equation and Equivalence

- FOC + Envelope Theorem, evaluated at $k_i = K$:

$$u'(F(K,1) + (1-\delta)K - G(K)) =$$

$$\beta u'(F(G(K),1) + (1-\delta)G(K) - G(G(K))) \cdot [1 - \delta + F_1(G(K),1)]$$

- This functional equation in G coincides with the planner's functional Euler equation from Chapter 4. Must be solved numerically unless $u = \log$, F Cobb-Douglas, $\delta = 1$.

Tractable Hetereogeneity

- With power utility, individual saving takes the form (proof on slide 38):

$$g(k_i, K) = \mu(K) + \lambda(K) k_i$$

- Saving is linear in own capital; slope $\lambda(K)$ is independent of k
- Wealth distribution does not affect aggregate saving $\Rightarrow$ not restrictive to use a representative agent
- With heterogeneous e_i: $g_i(k, K) = \mu_i(K) + \lambda(K)k$, with $\mu_i(K) = e_i\,\mu(K)$ (intercept proportional to e_i, slope common) (proof on slide 39) $\rightsquigarrow$ again, not restrictive to use a representative agent
- Note: agents are assumed to forecast with $K' = G(K)$; their optimal rules are then linear, so aggregate saving depends on K only, which confirms the assumption. This shows that an equilibrium with linear rules exists, not that it is unique

Aggregation: Proof (Guess and Verify)

- $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $R(K) \equiv 1 - \delta + r(K)$, $K' = G(K)$. Euler equation:

$$c(k_i, K)^{-\sigma} = \beta R(K')\, c(k_i', K')^{-\sigma} \;\Leftrightarrow\; c(k_i', K') = \left(\beta R(K')\right)^{1/\sigma} c(k_i, K)$$

- Guess consumption linear in own capital: $c(k_i, K) = a(K) + b(K) k_i$. The budget gives

$$k_i' = R(K) k_i + w(K) - c(k_i, K) = \underbrace{w(K) - a(K)}_{\mu(K)} + \underbrace{\left(R(K) - b(K)\right)}_{\lambda(K)} k_i$$

- Plug into the Euler equation; both sides are linear in k:

$$a(K') + b(K')\left[\mu(K) + \lambda(K) k_i\right] = \left(\beta R(K')\right)^{1/\sigma} \left[a(K) + b(K) k_i\right]$$

- Match coefficients (for all k_i):

$$\begin{aligned} k: &\quad b(K')\left(R(K) - b(K)\right) = \left(\beta R(K')\right)^{1/\sigma} b(K) \\ 1: &\quad a(K') + b(K')\left(w(K) - a(K)\right) = \left(\beta R(K')\right)^{1/\sigma} a(K) \end{aligned}$$

- These functional equations in $a(\cdot)$, $b(\cdot)$ involve K only, not k_i $\Rightarrow$ the guess is verified, and $g(k_i, K) = \mu(K) + \lambda(K) k$

Aggregation: Proof with Heterogeneous e_i

- Agent i's state is (k_i, e_i, K), with $\int_0^1 e_i\, di = 1$:

$$V(k_i, e_i, K) = \max_{k_i'} \{u(R(K)k_i + w(K)e_i - k_i') + \beta V(k_i', e_i, G(K))\}$$

  so the budget is $k_i' = R(K)k_i + w(K)e_i - c_i(k_i, K)$
- Guess $c_i(k_i, K) = a_i(K) + b_i(K)k_i$. Same steps as before give, for each i:

$$\begin{aligned} k_i: &\quad b_i(K')\,(R(K) - b_i(K)) = (\beta R(K'))^{1/\sigma}\, b_i(K) \\ 1: &\quad a_i(K') + b_i(K')\,(w(K)e_i - a_i(K)) = (\beta R(K'))^{1/\sigma}\, a_i(K) \end{aligned}$$

- The k_i-equation does not involve $e_i \Rightarrow$ same solution for all i: $b_i = b$, so $\lambda(K) = R(K) - b(K)$ is common
- The 1-equation is linear and homogeneous in $(a_i, e_i) \Rightarrow a_i(K) = e_i\, a(K)$, where a solves the equation with $e_i = 1$
- Hence

$$g_i(k_i, K) = \underbrace{e_i\,(w(K) - a(K))}_{\mu_i(K) = e_i \mu(K)} + \lambda(K) k_i$$

- Aggregating: $\int_0^1 g_i(k_i, K)\, di = \mu(K) \int_0^1 e_i\, di + \lambda(K) \int_0^1 k_i\, di = \mu(K) + \lambda(K)K = G(K)$, which is the consistency condition $G(K) = \int_0^1 g_i(k_i, K)\, di$ replacing $G(K) = g(K, K)$

Non Tractable Heterogeneity

- If $k_i \neq k_j$ and g is non-linear in k_i, aggregate saving $\int g(k_i, \cdot)\, di$ depends on the distribution of capital, not only on $K \rightsquigarrow$ aggregate state: distribution Γ, with $K(\Gamma) \equiv \int k\, d\Gamma(k)$

Definition 10 (Recursive CE, NGM, Inelastic Labour Supply, Non-Linear Decision Rules)

A recursive CE consists of functions $r(\Gamma)$, $w(\Gamma)$, $\mathcal{H}^\star(\Gamma)$, $V^\star(k_i, \Gamma)$, and $g^\star(k_i, \Gamma)$ such that

1. $V^\star(k_i, \Gamma)$ *solves*

$$V(k_i, \Gamma) = \max_{k_i'} \{u((1 - \delta + r(\Gamma))k_i + w(\Gamma) - k_i') + \beta V(k_i', \mathcal{H}^\star(\Gamma))\}$$

   for all (k_i, Γ), with $k_i' = g^\star(k_i, \Gamma)$ attaining the max
2. $r(\Gamma) = F_1(K(\Gamma), 1)$ *and* $w(\Gamma) = F_2(K(\Gamma), 1)$ *for all* Γ
3. *Consistency:* $\Gamma' = \mathcal{H}^\star(\Gamma)$ *is the distribution of* $k' = g^\star(k, \Gamma)$ *when* $k \sim \Gamma$

- Implied aggregate: $K' = \int g^\star(k, \Gamma)\, d\Gamma(k) \neq g^\star(K, K)$ in general

Recursive CE with Valued Leisure

- Assume agents have utility functions $u(c) - v(\ell)$

Definition 11 (Recursive CE with Valued Leisure, Representative Agent case)

A recursive CE consists of $r(K)$, $w(K)$, $G^\star(K)$, $H^\star(K)$, $V^\star(k_i, K)$, $g^\star(k_i, K)$, $h^\star(k_i, K)$ *such that*

1. $V^\star(k_i, K)$ *solves*

$$V(k_i, K) = \max_{k_i', \ell} \{u((1 - \delta + r(K))k_i + w(K)\ell - k_i') - v(\ell) + \beta V(k_i', G^\star(K))\}$$

   for all (k_i, K)*, with* $g^\star$*,* $h^\star$ *attaining the max*
2. $r(K) = F_1(K, H^\star(K))$ *and* $w(K) = F_2(K, H^\star(K))$ *for all* K
3. $G^\star(K) = g^\star(K, K)$ *and* $H^\star(K) = h^\star(K, K)$ *for all* K

- The agent does not use $H^\star$ directly; $G^\star$ suffices because r and w capture how labor input changes with K. Aggregation breaks unless $k_{i,0}/e_i$ is equalized across agents: otherwise the aggregate state becomes the distribution Γ of (k, e) (next slide).

Valued Leisure without Aggregation

- If $k_{i,0}/e_i$ differs across agents, K is no longer a sufficient state $\rightsquigarrow$ aggregate state: joint distribution Γ of (k, e), with $K(\Gamma) \equiv \int k \, d\Gamma(k, e)$

Definition 12 (Recursive CE with Valued Leisure, Heterogeneous Agents)

A recursive CE consists of $r(\Gamma)$, $w(\Gamma)$, $\mathcal{H}^\star(\Gamma)$, $L^\star(\Gamma)$, $V^\star(k_i, e_i, \Gamma)$, $g^\star(k_i, e_i, \Gamma)$, $h^\star(k_i, e_i, \Gamma)$ *such that*

1. $V^\star(k_i, e_i, \Gamma)$ *solves*

$$V(k_i, e_i, \Gamma) = \max_{k_i', \ell} \{u((1 - \delta + r(\Gamma))k_i + w(\Gamma)e_i\ell - k_i') - v(\ell) + \beta V(k_i', e_i, \mathcal{H}^\star(\Gamma))\}$$

   for all (k_i, e_i, Γ)*, with* $g^\star$*,* $h^\star$ *attaining the max*
2. $r(\Gamma) = F_1(K(\Gamma), L^\star(\Gamma))$ *and* $w(\Gamma) = F_2(K(\Gamma), L^\star(\Gamma))$ *for all* Γ
3. *Consistency: for all* Γ, $L^\star(\Gamma) = \int e \, h^\star(k, e, \Gamma) \, d\Gamma(k, e)$ *and* $\Gamma' = \mathcal{H}^\star(\Gamma)$ *is the distribution of* $(k', e) = (g^\star(k, e, \Gamma), e)$ *when* $(k, e) \sim \Gamma$

- e_i is fixed over time: only the capital dimension of Γ moves
- Implied aggregates $K' = \int g^\star(k, e, \Gamma) \, d\Gamma(k, e)$ and $L = L^\star(\Gamma)$ depend on the whole Γ, not only on K: no representative agent

5. Overlapping Generations

OLG: Setup

- Agents live two periods (y = young, o = old); no altruism, no bequests
- Cohort t utility: $u_t(c_y, c_o) = u(c_y) + \beta u(c_o)$
- Initial old (generation -1): $u_{-1}(c) = u(c)$
- Endowments: $(\omega_{y,t}, \omega_{o,t+1})$ for cohort t
- **Key differences from dynastic models**:
  - × Simpler maximization (finite horizon)
  - × First-order dynamics (not second-order)
  - × Equilibria can be Pareto inefficient
  - × Negative real interest rates possible

Sequential CE, OLG Endowment

Definition 13 (Sequential CE, OLG Endowment)

A sequential CE is $\{c^\star_{i,t}\}_{t=0}^\infty$ *for each* $i \in \{y, o\}$, $\{a^\star_{t+1}\}_{t=0}^\infty$, *and* $\{q_t\}_{t=0}^\infty$ *such that*

1. *for each* $t > 0$, $(c^\star_{y,t}, c^\star_{o,t+1}, a^\star_{t+1})$ *solves*

$$\max_{c_{y,t}, c_{o,t+1}, a_{t+1}} u(c_{y,t}) + \beta u(c_{o,t+1})$$

$$\text{s.t.} \quad c_{y,t} + q_t a_{t+1} = \omega_{y,t} \quad \text{and} \quad c_{o,t+1} = \omega_{o,t+1} + a_{t+1}$$

   and $c^\star_{o,0} = \omega_{o,0}$

2. *for all* $t \geq 0$, $c^\star_{y,t} + c^\star_{o,t} = \omega_{y,t} + \omega_{o,t}$

- Goods market clearing $\Leftrightarrow a^\star_{t+1} = 0$ for all t: any saving by the young requires someone on the other side, but the old are in their last period and the next cohort is unborn.

Characterization of Sequential CE, OLG Endowment

- Since $a^\star_{t+1} = 0$: **autarky**, $(c^\star_{y,t}, c^\star_{o,t}) = (\omega_{y,t}, \omega_{o,t})$.
- Equilibrium price from the Euler equation at autarky (CRRA utility):

$$q_t = \beta \left(\frac{\omega_{y,t}}{\omega_{o,t+1}} \right)^\sigma$$

- If $\omega_y > \omega_o$ (income declines over the life cycle, but not with time): $q > 1$ possible $\Rightarrow$ **negative real interest rate**. Cannot happen in the dynastic model (unless aggregate endowments decline with time).
- **Intuition**: No smoothing feasible (old cannot trade with unborn). Transferring from young to old (e.g. PAYG pension) can make all generations better off $\Rightarrow$ market outcome is inefficient. Chapter 6 analyzes this.

Sequential CE, OLG Growth Economy

Definition 14 (Sequential CE, OLG Growth Economy)

Labor productivities e_y, e_o with $e_y + e_o = 1$; stationary $F(k, \ell)$.
A sequential CE is $\{c^\star_{i,t}\}_{t=0}^\infty$ for each $i \in \{y, o\}$, $\{k^\star_{t+1}\}_{t=0}^\infty$, $\{r_t\}_{t=0}^\infty$, $\{w_t\}_{t=0}^\infty$ such that

1. *for each $t > 0$, $(c^\star_{y,t}, c^\star_{o,t+1}, k^\star_{t+1})$ solves*

$$\max_{c_{y,t}, c_{o,t+1}, k'} u(c_{y,t}) + \beta u(c_{o,t+1})$$

$$\text{s.t.} \quad c_{y,t} + k_{t+1} = w_t e_y \quad \text{and} \quad c_{o,t+1} = w_{t+1} e_o + (1 - \delta + r_{t+1}) k_{t+1}$$

   and $c^\star_{o,0} = w_0 e_o + (1 - \delta + r_0) k_0$

2. *for all t, $r_t = F_1(k^\star_t, 1)$ and $w_t = F_2(k^\star_t, 1)$, with $k^\star_0 \equiv k_0$*
3. *for all $t \geq 0$, $c^\star_{y,t} + c^\star_{o,t} + k^\star_{t+1} = F(k^\star_t, 1) + (1 - \delta) k^\star_t$*

- Substituting prices as functions of capital in the Euler equation:

$$u'(e_y F_2(k_t, 1) - k_{t+1}) =$$

$$\beta u'(e_o F_2(k_{t+1}, 1) + (1 - \delta + F_1(k_{t+1}, 1))k_{t+1}) \cdot (1 - \delta + F_1(k_{t+1}, 1))$$

- **First-order** difference equation: given k_t, solve for k_{t+1} and iterate forward
- Dynastic model: second-order (needs two boundary conditions)
- In general, $\beta(1 - \delta + F_1(\bar{k}, 1)) \neq 1$: the interest rate depends on life-cycle income
- No immediate connection to a planner's problem $\Rightarrow$ uniqueness not guaranteed

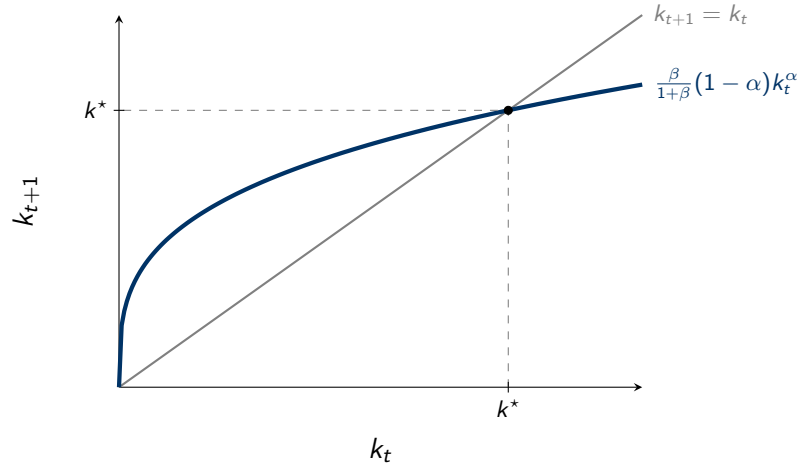
OLG: Example

- With $u = \log$, $e_y = 1$, $e_o = 0$ (retirement), $F = K^\alpha L^{1-\alpha}$, $\delta = 1$:

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)(k_t)^\alpha$$

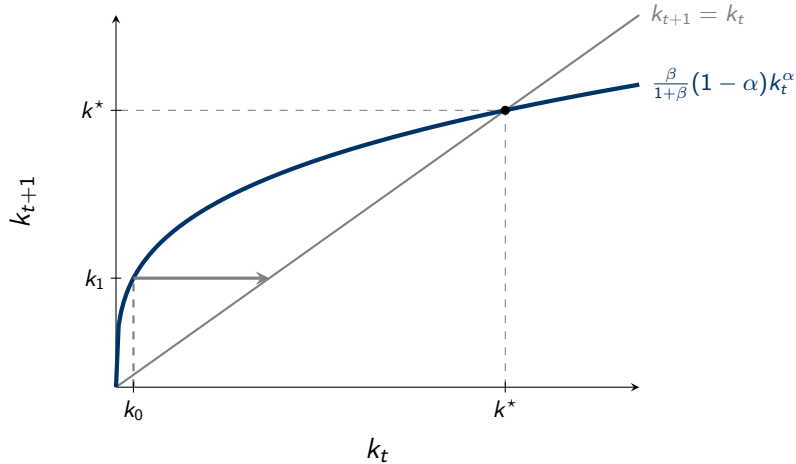
- Log-linear dynamics, monotonic convergence to steady state
- Steady-state gross interest rate: $\alpha(1+\beta)/(1-\alpha)$, which can be < 1 if α is low enough

OLG: Dynamics



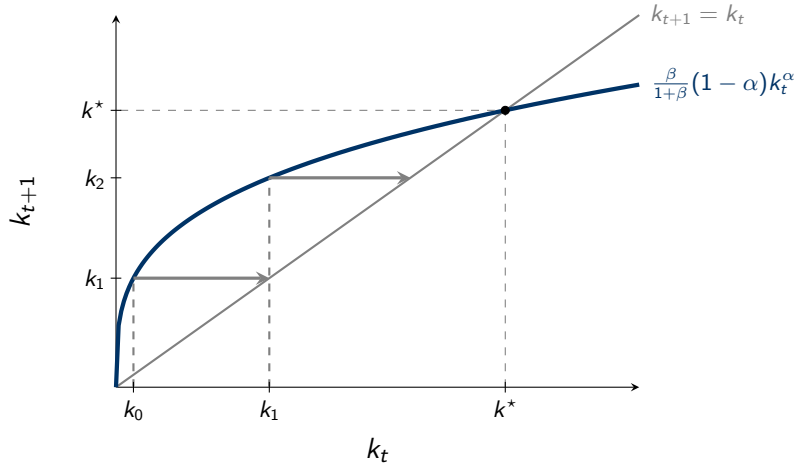
Example with $\alpha = 0.3$, $\beta = 1$: starting from $k_0 < k^\star$, capital rises monotonically to $k^\star = \left[\frac{\beta}{1+\beta}(1-\alpha)\right]^{\frac{1}{1-\alpha}}$.

OLG: Dynamics



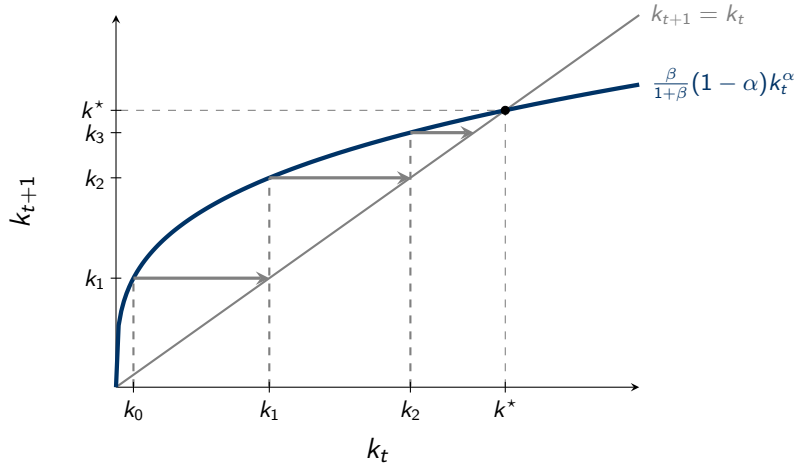
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OLG: Dynamics



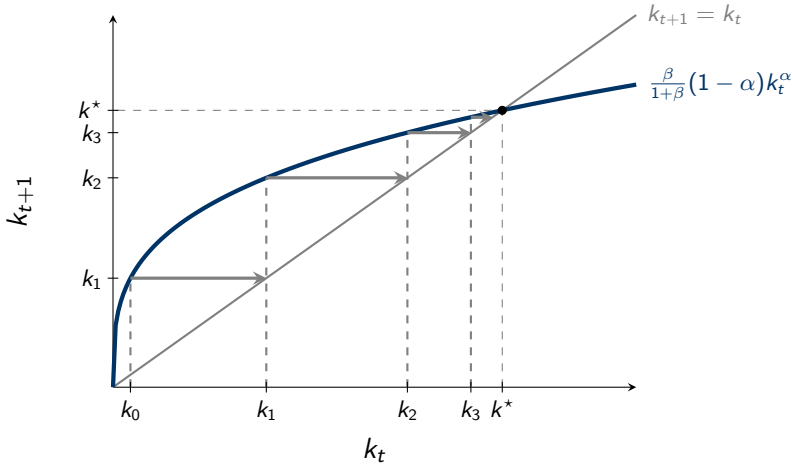
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OLG: Dynamics



Example with $\alpha = 0.3$, $\beta = 1$: starting from $k_0 < k^\star$, capital rises monotonically to $k^\star = \left[\frac{\beta}{1+\beta}(1-\alpha)\right]^{\frac{1}{1-\alpha}}$.

OLG: Dynamics



Example with $\alpha = 0.3$, $\beta = 1$: starting from $k_0 < k^\star$, capital rises monotonically to $k^\star = \left[\frac{\beta}{1+\beta}(1-\alpha)\right]^{\frac{1}{1-\alpha}}$.

- **Perpetual youth**: All individuals face a constant probability of death each period. As survival probability $\to 1$: approaches the dynastic model.
- **Warm glow**: $u(c_y) + \beta u(c_o) + \varphi(b')$, where φ is utility from the act of giving (behavioral model).
- **Pure altruism**: Parent values child's utility as the child does:

$$\varphi(b) = \max_{c_y, c_o, a', b'} \left\{ u(c_y) + \beta u(c_o) + \tilde{\beta} \varphi(b') \right\}$$

subject to budget constraints. Recursive structure $\Rightarrow$ dynastic life-cycle model with long-run interest rate $= 1/\tilde{\beta}$.

6. Summary

Key Takeaways (I)

1. **Three formulations**: Arrow-Debreu (Defs 1–4), sequential (Defs 5–6, 13–14), recursive (Defs 7–12). All yield the same allocations in dynastic economies.
2. **First Welfare Theorem**: With identical agents and no frictions, CE = planner's solution.
3. **Aggregation**: Power utility $\Rightarrow g(k, K) = \mu(K) + \lambda(K)k \Rightarrow$ wealth distribution irrelevant. Breaks with valued leisure.
4. **Investment optimality**: $p_t = p_{t+1}[r_{t+1} + 1 - \delta]$.

Key Takeaways (II)

5. **OLG**: First-order dynamics; interest rate depends on life-cycle income; negative rates possible; can be Pareto inefficient.
6. **Pure altruism**: OLG $\rightarrow$ dynasty as special case; long-run $r = 1/\tilde{\beta}$.
7. **Departures from perfect competition** (frictions, monopoly, incomplete markets) $\Rightarrow$ CE $\neq$ planner's solution. Chapters 6 and beyond.

