2026-2027 – Econ 0107 – Macroeconomics I

Chapter 6 : Welfare

(Chapter 6 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook)

Franck Portier
f.portier@ucl.ac.uk

University College London

Version 1.0
02/11/2026

Outline

1. The First Welfare Theorem
2. Tracing Out the Pareto Frontier
3. Inefficient Market Outcomes
4. Overlapping Generations: Efficiency
5. Optimal Government Policy
6. Summary

1. The First Welfare Theorem

Abstract Setup

- Finite set I of consumers, finite set J of firms. Vectors of goods x, y, ω.
- Resource constraint: $\sum_{i \in I} x_i \leq \sum_{j \in J} y_j + \sum_{i \in I} \omega_i$
- Notation:
  - X_i: consumption possibility set for consumer i
  - Y_j: production possibility set for firm j
  - $\succsim_i$: preference ordering for consumer i
  - $\theta_{i,j}$: consumer i's ownership share of firm j, with $\sum_i \theta_{i,j} = 1$ for all j
  - p: price vector
- **Example**: static model, 3 goods (consumption good, capital, labour), one consumer with endowments of capital and labour $(k, 1)$, one firm with CRS production $y \leq F(k, \ell)$.

  $$x = (c, 0, 0) \in \mathbb{R}^3_+, \quad y = (y, -k, -\ell) \in \mathbb{R}_+ \times \mathbb{R}^2_-, \quad \omega = (0, k, 1), \quad p = (1, r, w)$$

- **Key assumption**: Local non-satiation (LNS) – each consumer can always be made better off by an infinitesimal change in consumption.

Competitive Equilibrium (Abstract)

- A competitive equilibrium is a consumption allocation $\{x_i^\star\}_{\forall i}$, a production allocation $\{y_j^\star\}_{\forall j}$, and a price system $p^\star$ such that
  1. for each $i \in I$, $x_i^\star$ is in X_i and there is no $x \in X_i$ such that $x \succ_i x_i^\star$ and

$$px \leq px_i^\star = p\omega_i + \sum_j \theta_{i,j} py_j$$

  2. for each $j \in J$, $y_j^\star \in Y_j$ and there is no $y \in Y_j$ such that $py > py_j^\star$
  3. market clearing: the resource constraint holds with equality

Theorem 6.1 (First Welfare Theorem)

Theorem 1

An allocation that is part of a competitive equilibrium is Pareto optimal.

Proof (by contradiction):

- Suppose $\{\tilde{x}_i\}_{\forall i}$, $\{\tilde{y}_j\}_{\forall j}$ is feasible and Pareto dominates the equilibrium:

$$\forall i \in I : \tilde{x}_i \succsim_i x_i^\star, \qquad \exists \tilde{\imath} \in I : \tilde{x}_{\tilde{\imath}} \succ_{\tilde{\imath}} x_{\tilde{\imath}}^\star$$

- From consumer optimization + LNS: $p\tilde{x}_i \geq px_i^\star$ for all i, with strict inequality for $\tilde{\imath}$.
- Summing household budgets:

$$\sum_{i \in I} p\tilde{x}_i > p\omega + \sum_{j \in J} py_j^\star$$

Theorem 6.1 (First Welfare Theorem): Proof (cont.)

- From profit maximization: $p\tilde{y}_j \leq py_j^\star$ for all j. Therefore:

$$\sum_{i \in I} p\tilde{x}_i > p\omega + \sum_{j \in J} p\tilde{y}_j$$

- But feasibility requires

$$\sum_{i} p\tilde{x}_i \leq p\omega + \sum_{j} p\tilde{y}_j$$

- Contradiction. ■

Mapping to Macroeconomic Models

- **Static model**: One consumer with endowments $(k, 1)$, one firm with CRS production.

$$x \in \mathbb{R}^3_+, \quad y = (y, -k, -\ell) \in \mathbb{R}_+ \times \mathbb{R}^2_-, \quad \omega = (0, k, 1), \quad p = (1, r, w)$$

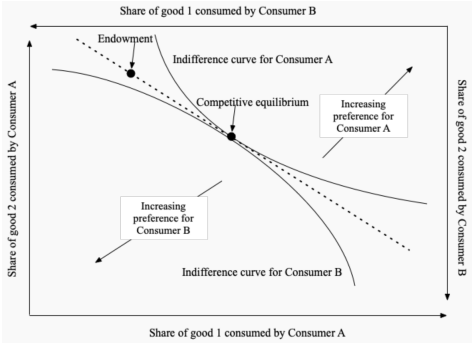
- **Dynamic model with** $T < \infty$: Goods at different dates are additional market goods. Vectors become T times longer.
- **Infinite horizon**: Vectors are infinite-dimensional. The proof carries through provided total expenditures are finite:

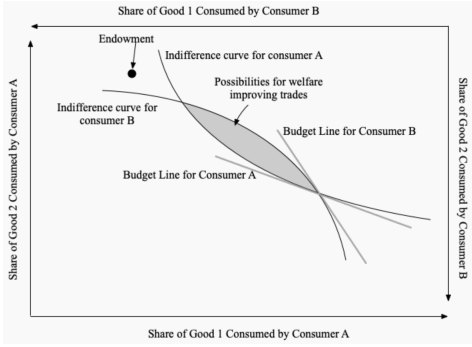
$$\sum_{t=0}^{\infty} p_t c_{i,t} < \infty$$

  × Typically satisfied in dynastic models (p_t falls at rate $\beta < 1$). Can fail in OLG models.

Intuition: Edgeworth Box

Endowment economy with two goods and two consumers





- **Panel (a)** (efficient equilibrium): Both consumers face the same prices. Indifference curves tangent to the budget line and to each other $\Rightarrow$ no Pareto improvement possible.
- **Panel (b)** (distorted market): Consumers face different (after-tax) prices. Indifference curves intersect $\Rightarrow$ shaded region represents Pareto-improving reallocations.

2. Tracing Out the Pareto Frontier

The Social Planner's Problem

- A social planner maximizes a weighted sum of utilities:

$$\max_{\{c_{i,t}\}} \sum_{i \in I} \mu_i U_i, \qquad U_i = \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

  subject to the resource constraint $\sum_{i \in I} c_{i,t} \leq \sum_{i \in I} \omega_{i,t}$ for all t.

- The weights μ_i are called **Negishi weights** (or Pareto weights).

- Any solution to this problem must be Pareto optimal: if it were not, the planner could increase the objective.

Negishi Weights and Competitive Equilibrium

- Planner's FOC, where ψ_t is the multiplier on the resource constraint at t:

$$c_{i,t} = (u')^{-1}\left(\beta^{-t}\frac{1}{\mu_i}\psi_t\right)$$

- Competitive equilibrium FOC (from Chapter 5):

$$c_{i,t} = (u')^{-1}(\beta^{-t}\lambda_i p_t)$$

- Setting $\mu_i = 1/\lambda_i$ and $\psi_t = p_t$: the two allocations coincide.
- **Implication**: The CE is optimal for a planner who assigns higher weight to agents with higher date-0 wealth (λ_i lower $\Rightarrow$ μ_i higher).
- By varying $\{\mu_i\}$, we trace the **Pareto frontier**. The **Second Welfare Theorem** gives conditions (convexity) under which any Pareto-optimal allocation can be supported as a CE with appropriate lump-sum transfers.

3. Inefficient Market Outcomes

Taxes

- **Lump-sum taxes**: Do not affect any marginal conditions $\Rightarrow$ act as redistribution along the Pareto frontier. FWT holds.
- **Proportional labor tax** τ_ℓ: With valued leisure, the after-tax wage is $w_t(1-\tau_\ell)$. MRS between consumption and leisure $\neq$ MRT $\Rightarrow$ FWT fails.
  - $\times$ Exception: If leisure is not valued (inelastic labor), the tax is non-distortionary
- **Capital income tax** τ_k: After-tax return $r_t(1-\tau_k)$ enters the Euler equation. MRS between t and $t+1$ goods $\neq$ MRT $\Rightarrow$ FWT fails.
- In both cases, consumers and firms face different price vectors ($p^c \neq p^f$) $\Rightarrow$ the consumer inequality (at p^c) and the profit inequality (at p^f) can no longer be chained: the contradiction in the proof disappears.

Externalities: Competitive Equilibrium

- An externality arises when one agent's activity has payoff relevance to others, not reflected in prices.
- **Example**: Firm j produces $A(\bar{y})k_j^{\alpha}\ell_j^{1-\alpha}$, where A is decreasing in average output $\bar{y}$.
  - × A firm does not internalise that it reduces aggregate productivity.
  - × A planner would take that into account
  - × Therefore, competitive equilibrium is not efficient

Missing Markets: Borrowing Constraints

- Two-agent dynastic endowment economy, log utility.
- Endowments alternate: agent 1 has $2\omega/3$ in odd periods, $\omega/3$ in even periods; agent 2 reversed.
- **Unrestricted equilibrium**: Full consumption smoothing, $c_{1,t} = c_1$ and $c_{2,t} = c_2$ for all t, MRS$=\beta u'(c_{t+1})/u'(c_t) = q_t \rightsquigarrow$ Gross interest rate $= 1/\beta$.
- **With borrowing constraint** $a_{t+1} \geq 0$: Euler equation becomes an inequality when the borrowing constraint binds:

$$q_t u'(c_t) \geq \beta u'(c_{t+1}) \qquad (\star)$$

- **Result**: autarky. No smoothing. Clearly not Pareto optimal.

Borrowing Constraints: What goes wrong in the proof?

- As with externalities, the possibility sets X_i change:
  - × They contain *more restrictions*: one extra constraint per good, e.g. $c_0 \leq a_0 + \omega_0$, $q_0 c_1 \leq \omega_0 + a_0 - c_0 - q_0 \omega_1$, ...
  - × The equilibrium can be dominated by an allocation (e.g. full smoothing) that is simply *not in* the agent's consumption possibility set.
- The sets are also **endogenous**: they depend on prices (q_t appears in the constraints).
  - × Departs from the abstract setup, where X_i is exogenous.
  - × Interdependence: through prices, each agent's choices shape the sets the others face.

Lack of Commitment

- When agents can default on debt without punishment: leads to de-facto borrowing constraints (rational lenders refuse to lend).
- With punishment: intertemporal trade occurs, but the punishment mechanism itself must be committed to.
- **Examples studied later**:
  - × Government cannot commit to future tax policy (Chapter 15)
  - × Sovereign debt default (Chapter 24)

Market Power

- Some agents have **market power**: through their own behavior, they can affect the prices they transact at.
  - × Individual level: a worker may bargain for a higher salary (labor-market frictions, later chapters).
  - × Firm level: a firm sets the price of its product above marginal cost (**markups**), or bargains with input suppliers (growth and business cycles, later chapters).
- In macro models, market power is almost always assumed to be **limited**: it affects an individual transaction, not aggregates.
  - × **Monopolistic competition** (Dixit and Stiglitz, 1977): each firm has a monopoly over a specific good but competes with a continuum of imperfect substitutes, each negligible in the aggregate.
  - × Likewise, a worker has no influence on aggregates in wage negotiations.
  - × Exceptions exist (Apple is *granular* and may affect the whole smartphone market), but are likely rare.
- **FWT**: the abstract proof cannot be used directly, since some agents do not take p as given.

Quantifying Welfare Losses

- As an example, compare two allocations: (c_l, ℓ_l) under lump-sum taxes vs. (c_d, ℓ_d) under distortionary taxes.
- Define the **consumption-equivalent welfare loss** $\Delta > 0$:

$$u(c_l(1-\Delta), \ell_l) = u(c_d, \ell_d)$$

- Δ is the percentage reduction in consumption (at fixed hours) that makes the better allocation as good as the worse one.
  - × Has a real interpretation (unlike "utils")
  - × In dynamic models: Δ applies uniformly across all periods
  - × Alternative: Δ as a percentage *increase* in the d allocation needed to reach the l one (gives a different number)

4. Overlapping Generations: Efficiency

OLG Endowment Economy: Setup

- Representative consumer per cohort, two-period lives, $u_t(c_y, c_o) = \log c_y + \log c_o$.
- Old of generation -1: $u_{-1} = \log c$
- Stationary endowments (ω_y, ω_o).

- Competitive equilibrium: sequential trading, no borrowing constraints.

- Consumer born at t solves:

$$\max_{c_y, c_o} \log c_y + \log c_o \quad \text{s.t.} \quad c_y + q_t c_o = \omega_y + q_t \omega_o$$

- Generation t consumer cannot do intertemporal trade (no other generation lives in t and $t+1$).
- **Equilibrium**: autarky, $c_{y,t} = \omega_y$, $c_{o,t} = \omega_o$, bond price $q_t = \omega_y / \omega_o$ (from Euler equation).

OLG Endowment Economy: Is the Equilibrium Efficient?

- **Case 1 ("incentives to save")**: $\omega_y = 3$, $\omega_o = 1$. Bond price $q = 3$ (gross interest rate $1/3$, interest rate = -66%).
  - × Alternative allocation: $\tilde{c}_{y,t} = \tilde{c}_{o,t} = 2$ for all t (transfer 1 from young to old each period) (it is feasible).
  - × Initial old: strictly better ($2 > 1$)
  - × All subsequent generations: $\log 2 + \log 2 = \log 4 > \log 3 + \log 1 = \log 3 \Rightarrow$ strictly better
  - × $\Rightarrow$ Equilibrium (autarky) is **not** Pareto optimal.

- **Case 2 ("incentives to borrow")**: $\omega_y = 1$, $\omega_o = 3$. Bond price $q = 1/3$ (gross interest rate 3, interest rate = +200%).
  - × The $(2, 2)$ alternative improves all generations born at $t \geq 0$ but makes the initial old worse off ($2 < 3$).
  - × No other allocation can Pareto dominate $\Rightarrow$ equilibrium (autarky) **is** Pareto optimal.

Revisiting the Proof of the FWT in OLG

- In the OLG model, equilibria are sometimes Pareto optimal and sometimes not. Why?
- Go back to the abstract proof of the FWT: the notation can include infinite sequences, and the proof goes through in all parts, except possibly one:
  - × when summing the budgets over all agents, I is now **infinite**.
- Prices are the sequence $p = \{p_0, p_1, p_2, \ldots\}$, with $p_t = \prod_{\tau=0}^{t-1} q_\tau$.
- $\omega_y = 3$, $\omega_o = 1$: $q_t = 3$ **for all** t: the present value of the equilibrium allocation is

$$\sum_{t=0}^{\infty} p_t(\omega_y + \omega_o) = (\omega_y + \omega_o)\left(1 + 3 + 3^2 + \cdots\right) = \infty$$

  - × This proof strategy cannot be used (indeed, the $(2, 2)$ allocation Pareto dominates the equilibrium).
- $\omega_y = 1$, $\omega_o = 3$: $q_t = 1/3$ **for all** t: the present value is finite, and the proof does go through.
  - × We in fact have a proof that no allocation can Pareto improve on the autarkic allocation $(c_{y,t}, c_{o,t}) = (1, 3)$ for all t.

The Balasko-Shell Theorem (Endowment Economy)

Theorem 2 (Balasko and Shell, 1980)

A competitive equilibrium in an OLG endowment economy is Pareto optimal if and only if

$$\sum_{t=0}^{\infty} \frac{1}{p_t} = \infty$$

where $p_t = \prod_{\tau=0}^{t-1} q_\tau$ *is the price of consumption at* t *(the inverse of the gross interest rate between 0 and* t*).*

- **Implication**: When the gross real interest rate converges, the equilibrium is efficient if and only if the limit net real interest rate is non-negative.
- With growth at rate g: efficient if and only if the limit interest rate $\geq g$.
- When the equilibrium is inefficient, a PAYG transfer scheme (young $\rightarrow$ old) makes all generations better off. But this requires government commitment to an infinite sequence of transfers.

Dynamic Efficiency with Production

- Neoclassical production economy with OLG.
- (ω_y, ω_o) is the endowments of a cohort in labor efficiency units.

$$c_y + s = \omega_y w_t, \qquad c_o = s(1 - \delta + r_{t+1}) + \omega_o w_{t+1}$$

- Asset market clearing: $s_t = k_{t+1}$. Competitive pricing: $r_t = F_1(k_t, \omega_y + \omega_o)$, $w_t = F_2(k_t, \omega_y + \omega_o)$.

Dynamic (In)Efficiency

- **Dynamic (in)efficiency** is a *production* concept about the capital path $\{k_t\}$:
- Dynamically inefficient: another feasible path $\{k_t\}$ gives at least as much aggregate consumption at every date, strictly more at some date.
- In the standard Diamond OLG model: the equilibrium is Pareto optimal iff it is dynamically efficient.

Theorem 3

Define $R_t = 1 - \delta + F_1(k_t, \omega_t)$. *Then* $\{k_t\}_{t=0}^{\infty}$ *is dynamically efficient if and only if*

$$\sum_{t=0}^{\infty} \prod_{s=1}^{t} R_s(k_s) = \infty$$

- **Steady state**: efficient iff $R = F_1(k_{ss}, \omega) + 1 - \delta \geq 1$, i.e., the net interest rate is non-negative.

Dynamic Inefficiency: Intuition

- **When $R < 1$ at steady state**:
  - × Reduce saving by ε at t. Resources freed for consumption at t.
  - × At $t+1$, production falls by less than ε (since marginal return < 1).
  - × Reduce saving by ε again at $t+1$: net resources freed for consumption because production has fallen by less than ε.
  - × Repeat forever $\Rightarrow$ **free lunch** at all dates.

- **When $R \geq 1$**:
  - × Reducing saving by ε at t reduces resources at $t+1$ by more than ε.
  - × Future reductions grow and eventually become infeasible.

Dynamic Inefficiency: Oversaving

- **Oversaving**: In OLG models, equilibrium savings can be "too high" because each generation saves for retirement without regard for the aggregate capital stock.
- In such a case of dynamic inefficiency, rational bubbles can be sustained and they are Pareto improving. They are a mean of reducing investment in k.

- Fundamentally different from the dynastic model, where the TVC prevents over-accumulation.

- The warm-glow model (Chapter 5) can also generate dynamic inefficiency: when $v(b) = Ab$ (linear), convergence to a steady state with $r - \delta < 0$.

5. Optimal Government Policy

Correcting Market Failures

- **Externalities** (PIGOU, 1920): A tax equal to the marginal external damage, evaluated at the efficient allocation, restores optimality.
  - × Alternative: assign property rights (COASE) and let the market internalize the externality. Not always feasible.
- **Monopoly**: Subsidize production at a per-unit rate to counteract under-production. Alternatively, anti-trust regulation.
  - × But monopoly power can incentivize innovation through patents ⇒ trade-off (Chapter 13).
- **Distortionary taxes**: Compare alternative feasible tax systems (Ramsey analysis, Chapter 15). Which tax base is least distortionary?
- **Missing markets**: Beware the "chicken model" – if a market is missing, it is usually missing for a reason (moral hazard, adverse selection).

Redistribution

- A separate aim from efficiency: achieving a more equitable distribution, even if some agents are made worse off.
- Common approach: utilitarian social welfare function with equal weights, where ω_i is the fraction of type-i agents:

$$\max \sum_i \omega_i u(c_i) \quad \text{s.t.} \quad \sum_i \omega_i c_i = \sum_i \pi_i \omega_i$$

- With non-distortionary redistribution: equal weights $\Rightarrow$ equal consumption. With distortionary taxes: optimal redistribution is partial.
- **Behind the veil of ignorance**: Before knowing one's type, an agent would choose the utilitarian optimum as an insurance scheme.
  - $\times$ Cannot be offered by markets (agents not yet born), but government can implement it.

6. Summary

Key Takeaways (I)

- **First Welfare Theorem**: Under LNS, a competitive equilibrium is Pareto optimal.
  - × Proof: the alternative allocation must be more expensive (consumer optimization) but cannot be (profit maximization + feasibility). Contradiction.
- **Negishi weights**: CE is optimal for a planner with $\mu_i = 1/\lambda_i$. Varying weights traces the Pareto frontier.
  - × Second Welfare Theorem: any Pareto optimum can be decentralized with lump-sum transfers (under convexity).
- **Market failures**:
  - × Distortionary taxes (consumers and firms face different prices)
  - × Externalities (possibility sets interdependent)
  - × Missing markets (additional constraints, pecuniary externalities)
  - × Lack of commitment
  - × Monopoly power (price-setting, not price-taking)

Key Takeaways (II)

- **OLG efficiency**: Even without frictions, OLG equilibria can be Pareto inefficient.
  - × Balasko-Shell: efficient iff $\sum 1/p_t = \infty$, i.e., the asymptotic net interest rate ≥ 0 (or $\geq g$ with growth).
  - × Dynamic inefficiency = oversaving.
- **Welfare measurement**: Consumption-equivalent Δ – the percentage reduction in consumption that equates utility across allocations.
- **Policy**: Pigou taxes for externalities, Ramsey analysis for tax design, Mirrlees for informational frictions. Beware the "chicken model."

