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Econ 0107 – Macroeconomics I – Franck Portier

Problem Set: A Framework for Macroeconomics

Chapter 2 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook

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Readings.
Required: Krusell, *Real Macroeconomic Theory*, Chapter 2, §§2.1–2.6 (the resource constraint and the aggregate production function; growth facts and balanced growth; the time horizon, population, preferences, and choice).

Background / classic references (optional):

- R. Solow (1956), "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*.
- N. Kaldor (1961), "Capital Accumulation and Economic Growth," in Lutz and Hague (eds.), *The Theory of Capital*.
- H. Uzawa (1961), "Neutral Inventions and the Stability of Growth Equilibrium," *Review of Economic Studies*.
- R. King, C. Plosser, and S. Rebelo (1988), "Production, Growth and Business Cycles I: The Basic Neoclassical Model," *Journal of Monetary Economics*.

Problem 1 – The aggregate production function and factor income shares

Let $F(K, L)$ be the aggregate production function, assumed to display constant returns to scale (CRS), $F(\lambda K, \lambda L) = \lambda F(K, L)$ for all $\lambda > 0$, and to be strictly increasing and strictly concave in (K, L). Factor markets are competitive: a representative firm chooses $K, L \geq 0$ to maximise static profits $F(K, L) - rK - wL$, taking the rental rate of capital r and the wage w as given.

1. Differentiate the CRS identity with respect to λ at $\lambda = 1$ to obtain *Euler's theorem* for F. Differentiate it instead with respect to K to show that F_K (and, symmetrically, F_L) is homogeneous of degree zero in (K, L).

2. Derive the competitive-equilibrium factor prices from the firm's first-order conditions, and use Euler's theorem to show that equilibrium profits are exactly zero.

3. Define the capital and labour income shares $\sigma_K \equiv rK/F(K, L)$ and $\sigma_L \equiv wL/F(K, L)$. Show that $\sigma_K + \sigma_L = 1$.

4. Specialise to the Cobb-Douglas case $F(K, L) = BK^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$. Compute σ_K and show it is constant *for every* (K, L), not only along some particular path. Writing $F(K, L) = Lf(k)$ with $k \equiv K/L$, show more generally that $\sigma_K(k) = kf'(k)/f(k) = \alpha$ for all k is an ordinary differential equation whose only solution is $f(k) = Bk^{\alpha}$ – i.e. Cobb-Douglas is the *only* production function consistent with an exactly constant labour share (fact 6 in the text) at every point in time, not just asymptotically.

Problem 2 – Balanced growth and the direction of technical progress

The resource constraint is $C_t + I_t = Y_t$ and capital accumulates as $K_{t+1} = (1 - \delta)K_t + I_t$. Labour input $L_t = L$ is constant over time (fact 7). A *balanced growth path* (BGP) is a path along which Y_t, K_t, C_t, I_t all grow at constant (possibly different *a priori*) gross rates forever.

1. Suppose $a_t = Ag_a^t$, $b_t = Bg_b^t$ with $A, B > 0$ and $a_t + b_t = Cg_c^t$ for *all* $t = 0, 1, 2, \ldots$, with $C > 0$. Show that this forces $g_a = g_b = g_c$. [*Hint:* divide by g_c^t and use the linear independence of distinct geometric sequences, or examine the case $t = 0$ together with the first difference of the identity.] Apply this lemma to $C_t + I_t = Y_t$ and to $K_{t+1} = (1 - \delta)K_t + I_t$ to conclude that Y_t, C_t, I_t, K_t all grow, on a BGP, at one and the same common rate g.

2. Let output be produced with purely labour-augmenting technology, $Y_t = F(K_t, A_tL)$, $A_t = \gamma^t$, with F CRS and strictly concave in K. Suppose $K_t = k_0g^t$ and $Y_t = y_0g^t$ solve the model exactly for all t. Using CRS to write $Y_t = \gamma^t F(k_0(g/\gamma)^t, L)$, and the strict concavity of $F(\cdot, L)$ to argue that a strictly concave function can agree with a

given linear function at no more than two points, show that $g = \gamma$ is required (unless $g = \gamma$ trivially): the common growth rate of output and capital must equal the rate of labour-augmenting technical progress.

3. Now suppose instead that technology is purely *capital*-augmenting, $Y_t = F(B_t K_t, L)$, $B_t = \mu^t$ ($\mu > 1$), with L constant, and let F be CES: $F(K,L) = \left[aK^\rho + (1-a)L^\rho\right]^{1/\rho}$, $\rho \neq 0$ (i.e. not Cobb-Douglas). Suppose a BGP with $K_t = k_0 g^t$, $Y_t = y_0 g^t$ existed. Show that as $t \to \infty$, $Y_t \sim a^{1/\rho}(\mu g)^t k_0$, so consistency with $Y_t = y_0 g^t$ for all t requires $\mu = 1$ – i.e. with a non-Cobb-Douglas CRS technology, capital-augmenting progress is *inconsistent* with balanced growth (this is the content of Uzawa's (1961) theorem; a fully general proof is beyond the scope of this problem).

4. Show that when F *is* Cobb-Douglas, $F(K,L) = K^\alpha L^{1-\alpha}$, a capital-augmenting technology $B_t = \mu^t$ can be rewritten as an equivalent labour-augmenting technology: find $\tilde{A}_t$ such that $F(B_t K, L) = F(K, \tilde{A}_t L)$ for all K, L. Conclude that Cobb-Douglas is the one knife-edge case in which the *direction* of factor-augmenting technical change is irrelevant for balanced growth, reconciling Question 3 with the special role Cobb-Douglas already played in Problem 1.

Problem 3 – Stationary factor prices and shares along the balanced growth path

Technology is labour-augmenting, $Y_t = F(K_t, A_t L)$, $A_t = \gamma^t$, L constant, F CRS. Write $Y_t = A_t L f(\hat{k}_t)$ with $\hat{k}_t \equiv K_t/(A_t L)$ (capital per unit of effective labour) and $f(\hat{k}) \equiv F(\hat{k}, 1)$. Factor markets are competitive, so $r_t = F_K(K_t, A_t L)$ and $w_t = F_L(K_t, A_t L)$. From Problem 2, on a BGP $\hat{k}_t$ is constant, equal to some $\hat{k}^*$.

1. Show that $r_t = f'(\hat{k}_t)$, a function of $\hat{k}_t$ alone, and conclude that r_t is constant on the BGP – fact 5 (a stationary real interest rate).

2. Show that $w_t = A_t\left[f(\hat{k}_t) - \hat{k}_t f'(\hat{k}_t)\right]$, and conclude that w_t grows at exactly the rate γ of labour-augmenting progress – the same rate as output – which is fact 4 (the wage grows at the rate of output growth).

3. Using Questions 1–2 and Problem 1, show that the capital and labour income shares $\sigma_{K,t} = r_t K_t / Y_t$ and $\sigma_{L,t} = w_t L / Y_t$ are both constant along the BGP (fact 6), with $\sigma_{K,t} = \hat{k}^* f'(\hat{k}^*)/f(\hat{k}^*)$, *even if F is not Cobb-Douglas*.

4. Krusell's text notes that the labour share displays only "very minor fluctuations" even away from a hypothetical balanced path (e.g. over the business cycle, or during a transition with $\hat{k}_t \neq \hat{k}^*$). Explain, using the formula for $\sigma_K(\hat{k}) = \hat{k} f'(\hat{k})/f(\hat{k})$ from Question 3 and Problem 1.4, why this stronger empirical fact pushes macroeconomists to restrict F to Cobb-Douglas globally, whereas Questions 1–3 of this problem show that stationary factor prices and shares *on the BGP alone* require no such restriction.

Problem 4 – Preferences consistent with balanced consumption growth

A household chooses consumption in two adjacent periods $t, t+1$ to maximise $u(c_t) + \beta u(c_{t+1})$, where u is strictly increasing and strictly concave. Let R_{t+1} denote the gross real interest rate between t and $t+1$: the amount of period-$(t+1)$ goods obtainable by forgoing one unit of the period-t good.

1. Using the principle that an optimising household equates its marginal rate of substitution between c_t and c_{t+1} to the relative price R_{t+1}, derive the consumption *Euler equation* $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$.

2. Suppose the economy is on a path with $R_{t+1} = R$ constant and $c_{t+1}/c_t = g$ constant for every t (balanced consumption growth). Substitute $c_{t+1} = g c_t$ into the Euler equation and explain why consistency of the resulting relation with every value of c_t visited along the (ever-growing) path requires the ratio $u'(c)/u'(gc)$ to be *independent of the level c*.

3. Verify that $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, $\sigma \geq 0$ (log utility for $\sigma = 0$), satisfies $u'(c)/u'(gc) = g^\sigma$ for every $c > 0$ and every $g > 0$, and hence is consistent with balanced growth at rate g whenever $g^\sigma = \beta R$, i.e. $g = (\beta R)^{1/\sigma}$ (consistent with the elasticity of intertemporal substitution $1/\sigma$ found in the Chapter 4 problem set).

4. Let $u(c) = ac - \frac{b}{2}c^2$ (quadratic, $a, b > 0$, defined for $c < a/b$ so that $u' > 0$). Compute $u'(c)/u'(gc)$ for $g \neq 1$ and show it is *not* constant in c (e.g. compare its limits as $c \to 0$ and as $c \to a/b$). Conclude that, outside the CRRA class, balanced consumption growth is in general impossible to sustain, illustrating by counterexample why the "if and only if" result stated in the text singles out CRRA utility.

Problem 5 – Labour supply and the stationarity of hours worked

Consider the static (one-period) labour-leisure choice of §8.2.2 of the text: the household has a unit time endowment split between hours $n_t = 1 - l_t$ and leisure l_t, earns a real wage w_t per hour, and spends all labour income on consumption, $c_t = w_t(1 - l_t)$ (no saving). Preferences are $u(c, l) = \dfrac{(c\,g(l))^{1-\sigma} - 1}{1 - \sigma}$, $\sigma \geq 0$ (with $\log(cg(l))$ for $\sigma = 1$), where g is strictly increasing and $c\,g(l)$ is strictly quasiconcave.

1. Write the household's problem $\max_l u\big(w(1-l), l\big)$. Show that, since u is an increasing transformation of $x \equiv c\,g(l) = w(1-l)g(l)$, maximising u is equivalent to maximising x over l, and derive the first-order condition $(1-l)g'(l) = g(l)$.

2. Note that the wage w cancelled out of the first-order condition in Question 1. Explain why this implies that the optimal leisure choice l^* – and hence hours worked $n^* = 1 - l^*$ – is the *same for every* t, regardless of how w_t grows over time, rationalising fact 7 (stationary hours worked) for *any* $\sigma \geq 0$.

3. Specialise to $\sigma = 0$: $u(c, l) = \log c + \log g(l)$. Re-derive the first-order condition for l directly from this additively separable (in logs) form, and check it coincides with the one found in Question 1, confirming consistency with Problem 4's finding that log utility is the knife-edge CRRA case.

4. Now suppose instead preferences were additively separable in *levels*, $u(c, l) = \dfrac{c^{1-\sigma} - 1}{1 - \sigma} + v(l)$ for some strictly increasing, strictly concave v, with $\sigma \neq 1$ (this is *not* nested in the $c\,g(l)$ class). Show that the first-order condition for l is $v'(l) = w^{1-\sigma}(1-l)^{-\sigma}$, which generally ties l^* to the level of w whenever $\sigma \neq 1$ – so hours worked would drift as w_t grows along the BGP, violating fact 7. Show that at $\sigma = 1$ (log consumption utility) the wage term $w^{1-\sigma} = w^0 = 1$ drops out identically, so that log consumption utility delivers wage-invariant (hence stationary) labour supply for *any* increasing, concave $v(l)$ – not only for the multiplicative form of Questions 1–3.