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Econ 0107 – Macroeconomics I – Franck Portier

Problem Set: Dynamic Optimization

Chapter 4 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook

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Problem 1 – Two-period consumption-saving model

A representative agent lives for two periods ($t = 0, 1$). She receives wage income $w_0 > 0$ in period 0 and $w_1 > 0$ in period 1, and can save or borrow at the constant net interest rate $r > 0$. Let a_1 denote savings carried into period 1, with the borrowing constraint $a_1 \geq 0$ and the terminal condition $a_2 = 0$. Preferences are $u(c_0) + \beta u(c_1)$ with $u(c) = \log c$ and $0 < \beta < 1$.

1. Substitute the two period budget constraints to obtain the *lifetime budget constraint* and write the Lagrangian associated to the constrained maximisation problem.

2. Derive the Kuhn-Tucker first-order conditions. Assuming an interior solution ($a_1 > 0$), derive the Euler equation.

3. Suppose $\beta(1+r) = 1$. Show that optimal consumption is constant across periods, $c_0^* = c_1^*$, and find its value. Verify that the interior-solution assumption is satisfied when $w_0 > w_1/(1+r)$.

4. Now suppose $\beta(1+r) < 1$ and $w_1 > \beta(1+r)w_0$. Show that the borrowing constraint binds at the optimum, find (c_0^*, c_1^*), and compute the shadow price λ of the constraint.

Problem 2 – Backward induction in the finite NGM ($T = 2$)

Consider the social planner's problem in the neoclassical growth model (NGM) with $T = 2$ (periods $t = 0, 1, 2$, terminal condition $k_3 = 0$). Utility is logarithmic $u(c) = \log c$, the production function is Cobb-Douglas $f(k) = Ak^\alpha$ ($0 < \alpha < 1$, $A > 0$), and there is full depreciation ($\delta = 1$). The initial capital stock $k_0 > 0$ is given.

1. Write the planner's problem and derive the Euler equations for $t = 0$ and $t = 1$.

2. Use backward induction, starting from $t = 1$, to obtain explicit expressions for k_2 as a function of k_1, and for k_1 as a function of k_0.

3. Express the optimal saving rates $s_t \equiv k_{t+1}/(Ak_t^\alpha)$ for $t = 0$ and $t = 1$. Verify they match the general formula from the text: $s_t = \alpha\beta \dfrac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}$.

4. What do s_0 and s_1 converge to as $T \to \infty$? Interpret the result.

Problem 3 – CRRA utility and the elasticity of intertemporal substitution

Consider a representative agent with CRRA utility $u(c) = \dfrac{c^{1-\sigma} - 1}{1 - \sigma}$, $\sigma > 0$, $\sigma \neq 1$, facing a linear production technology $f_t(k_t) = R_t k_t$ (with time-varying gross return $R_t > 0$). The agent maximises $\sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to the sequence of budget constraints.

1. Show that the Euler equation between periods t and $t+1$ is $u'(c_t) = \beta R_{t+1}\, u'(c_{t+1})$.

2. By iterating the Euler equation, show that $\dfrac{c_{t+s}}{c_t} = \left(\beta^s R_{t,t+s}\right)^{1/\sigma}$ where $R_{t,t+s} \equiv R_{t+1} R_{t+2} \cdots R_{t+s}$.

3. Compute the elasticity of intertemporal substitution (EIS), defined as

$$\text{EIS} \equiv \frac{\partial \log(c_{t+s}/c_t)}{\partial \log R_{t,t+s}}.$$

4. Interpret the cases $\sigma < 1$, $\sigma = 1$, and $\sigma > 1$ in terms of income and substitution effects when $R_{t,t+s}$ rises by 1%.

Problem 4 – No Ponzi Game condition and the Transversality Condition

Consider an infinitely-lived agent facing the budget constraint $c_t + a_{t+1} = (1+r)a_t$ for all $t \geq 0$, with initial wealth $a_0 > 0$ and constant interest rate $r > 0$. The agent maximises $\sum_{t=0}^{\infty} \beta^t u(c_t)$ with u strictly increasing.

1. After T substitutions of successive budget constraints, show that

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = a_0(1+r) - \frac{a_{T+1}}{(1+r)^T}.$$

2. State the no-Ponzi-game (nPG) condition and derive the infinite-horizon lifetime budget constraint (LBC) by taking $T \to \infty$.

3. Define the transversality condition (TVC). Using the assumption that u is strictly increasing and assets yield no direct utility, argue that an optimising agent chooses $\lim_{T\to\infty} a_{T+1}/(1+r)^T = 0$, so the LBC holds with equality.

4. Explain clearly the conceptual difference between the nPG condition and the TVC. Which one is externally imposed and which one is an optimality condition?

Problem 5 – Infinite-horizon NGM: guess and verify

Consider the infinite-horizon social planner's problem in the NGM with $u(c) = \log c$, $f(k) = Ak^\alpha$, and full depreciation $\delta = 1$.

1. Write the Euler equation and show that the functional Euler equation (FEE) for the policy function $g(k) = k'$ is

$$\frac{1}{Ak^\alpha - g(k)} = \frac{\beta\alpha A\, g(k)^{\alpha-1}}{A\, g(k)^\alpha - g(g(k))} \quad \forall k.$$

2. Guess that $g(k) = sAk^\alpha$ for some constant s (linear saving rule). Substitute into the FEE and solve for s.

3. Find the steady-state capital stock $\bar{k}$ implied by this policy function.

4. Verify that the transversality condition holds for the solution found in Question 2.

Problem 6 – Value function iteration: the Bellman equation

Use the same parametric NGM as in Problem 5 ($u(c) = \log c$, $f(k) = Ak^\alpha$, $\delta = 1$, infinite horizon).

1. Write the Bellman equation for this problem.

2. Starting from the initial guess $V_0(k) = 0$ for all k, compute $V_1(k)$ and $V_2(k)$ analytically. Identify the implied policy function at each iteration.

3. Guess that the value function takes the form $V(k) = a + b\log k$ and substitute into the Bellman equation. Derive the system of equations that a and b must satisfy and solve for b.

4. Show that the policy function obtained from the fixed-point of the Bellman equation matches the solution found in Problem 5.

Problem 7 – Recursive formulation: the periodic NGM

Consider a variant of the NGM in which total factor productivity alternates deterministically: $A_t = A_h$ in even periods and $A_t = A_l$ in odd periods ($A_h > A_l > 0$). Output is $A_t f(k_t)$ with $f(k) = k^\alpha$, $\delta = 1$, $u(c) = \log c$, and $0 < \beta < 1$.

1. Explain why the standard single-state Bellman equation $V(k) = \max\{u(c) + \beta V(k')\}$ does *not* represent this problem correctly. What additional information must the state include?

2. Write the problem as two linked Bellman equations $V_h(k)$ and $V_l(k)$.

3. Guess $V_h(k) = a_h + b \log k$ and $V_l(k) = a_l + b \log k$ (same slope b). Derive b and the policy functions $g_h(k)$ (capital chosen in a high-TFP period) and $g_l(k)$ (capital chosen in a low-TFP period).

4. Show that the saving rate $k'/(A_i k^\alpha)$ is the same in both states and equals the constant $\alpha\beta$ from Problem 5. Provide an economic interpretation.

Problem 8 – Linearisation of the policy function

Consider the infinite-horizon NGM with CRRA utility $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, Cobb-Douglas production $f(k) = k^\alpha$ ($A = 1$), depreciation rate $\delta > 0$, discount factor β, and capital-income share α.

1. Find the steady-state pair $(\bar{k}, \bar{c})$ from the Euler equation and resource constraint evaluated at $k_{t+1} = k_t = \bar{k}$.

2. Using the formula from the text, show that the coefficient Θ in the linearised policy function satisfies

$$\Theta = \frac{\beta^2(1-\alpha)}{\sigma}\left(\frac{1}{\beta} - 1 + \delta\right)\left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right) - \delta\right).$$

3. The slope a_1 of the linearised policy function $g(k) \approx \bar{k}(1 - a_1) + a_1 k$ around $\bar{k}$ satisfies a quadratic. State the formula (equation (4.30) in the text) and explain why only the root in $(0, 1)$ corresponds to the stable policy function.

4. Compute $(\bar{k}, \bar{c})$, Θ, and a_1 numerically for $\alpha = 0.33$, $\beta = 0.99$, $\delta = 0.10$, and $\sigma = 2$.

Problem 9 – MATLAB: Value function iteration for the NGM

This problem asks you to solve the standard NGM numerically using value function iteration (VFI). Use the parameters $\alpha = 0.33$, $\beta = 0.99$, $\delta = 0.10$, $\sigma = 2$, $A = 1$ (same as Problem 8).

1. Compute the steady state $(\bar{k}, \bar{c})$ analytically and set up a capital grid with $n_k = 300$ equally-spaced points on $[0.5\bar{k},\ 1.5\bar{k}]$.

2. Write a MATLAB program implementing VFI. Pre-compute the full $n_k \times n_k$ matrix of feasible consumption values and the corresponding utility matrix. Iterate until the sup-norm distance between successive value functions is below 10^{-8}.

3. Plot (i) the converged value function $V(k)$ and (ii) the policy function $g(k)$ together with the 45-degree line, marking the steady state.

4. Estimate the numerical slope of g at $\bar{k}$ by a least-squares fit of $g(k)$ on k over the grid points within $\pm 10\%$ of $\bar{k}$, and compare it to the analytical a_1 from Problem 8. Why would a simple two-point difference $[g(k_{i+1}) - g(k_{i-1})]/(k_{i+1} - k_{i-1})$ at the grid point closest to $\bar{k}$ be unreliable?

Problem 10 – MATLAB: Finite-horizon saving rates and capital convergence

Return to the log/Cobb-Douglas/full-depreciation case: $u(c) = \log c$, $f(k) = Ak^\alpha$, $\delta = 1$, $A = 1$, $\alpha = 0.33$, $\beta = 0.99$. The analytical saving rate in a T-period economy is (from the text)

$$s_t(T) = \alpha\beta\,\frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}, \qquad t = 0, 1, \ldots, T-1.$$

The infinite-horizon saving rate is $s^{\infty} = \alpha\beta$.

1. Write a MATLAB function `saving_rates(T,alpha,beta)` that returns the vector $(s_0, s_1, \ldots, s_{T-1})$ using the formula above.

2. Plot the saving-rate profiles for $T \in \{5, 10, 20, 50\}$ on a single figure, together with the infinite-horizon rate s^{∞} as a horizontal dashed line.

3. Simulate the capital path $\{k_t\}$ starting from $k_0 = 0.5\bar{k}$ (where $\bar{k} = (\alpha\beta)^{1/(1-\alpha)}$) for $T \in \{5, 10, 20, 50, \infty\}$ and plot all paths together.

4. For the infinite-horizon case, compute the *half-life* of convergence: the number of periods τ^* such that $|k_{\tau^*} - \bar{k}| \leq \frac{1}{2}|k_0 - \bar{k}|$. Comment on how it relates to the slope α of the policy function.