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Econ 0107 – Macroeconomics I – Franck Portier

Problem Set: Competitive Equilibrium in Dynamic Economies

Chapter 5 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook

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Problem 1 – Arrow–Debreu equilibrium in an endowment economy

A unit continuum of infinitely-lived agents $i \in [0,1]$ each maximises $\sum_{t=0}^{\infty} \beta^t \log c_{i,t}$ ($\beta \in (0,1)$) subject to the Arrow–Debreu (AD) budget constraint $\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t y_{i,t}$. Individual endowments are $y_{i,t} = e_i Y_t$ where $Y_t = Y_0 g^t$ ($g > 0$), and $\int_0^1 e_i \, di = 1$. Agents hold no initial assets. Normalise $p_0 = 1$.

1. Define the Arrow–Debreu competitive equilibrium for this economy. Derive the household's optimality condition relating p_t to the Lagrange multiplier λ_i and the marginal utility of consumption.

2. Using goods-market clearing $\int_0^1 c_{i,t} \, di = Y_t$ for all t, show that the equilibrium AD prices are $p_t = \beta^t g^{-t}$.

3. Derive the optimal consumption allocation and show that $c^*_{i,t} = e_i Y_t$ (the *consumption-sharing rule*). Does this equilibrium require any trade between agents? Explain.

4. Compute the present value of agent i's endowment stream, $W_i = \sum_{t=0}^{\infty} p_t y_{i,t}$. Show that W_i does not depend on the aggregate growth rate g, and give an economic interpretation.

Problem 2 – CRRA utility and the equilibrium real interest rate

Use the same economy as Problem 1, but replace the utility function with CRRA: $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$, $\sigma > 0$, $\sigma \neq 1$.

1. Using the household's optimality condition $u'(c_{i,t}) = \lambda_i p_t / \beta^t$ and goods-market clearing, show that the equilibrium AD prices are $p_t = \beta^t g^{-\sigma t}$ and that the consumption-sharing rule $c^*_{i,t} = e_i Y_t$ continues to hold.

2. Consider the sequential equilibrium with gross interest rate $1 + r_{t+1} = 1/q_t$ (where $q_t = p_{t+1}/p_t$ is the bond price). Show that the equilibrium interest rate is constant and equal to

$$1 + r_t = \frac{g^\sigma}{\beta} \quad \forall\, t \geq 1.$$

3. Interpret the *elasticity of intertemporal substitution* EIS $= 1/\sigma$. When $g > 1$, what is the sign of $\partial(1 + r_t)/\partial\sigma$? Which effect (income or substitution) dominates when $\sigma > 1$?

4. Suppose the economy is stationary: $g = 1$. Show that the real interest rate is $1 + r = 1/\beta$ regardless of σ. Now suppose g increases slightly above 1. Show that the interest rate rises by more the larger is σ. Give economic intuition.

Problem 3 – Arrow–Debreu equilibrium with linear production

A unit continuum of competitive firms produce output using labour: $Y_t = A_t L_t$, $A_t = A_0 g^t > 0$. Agent i supplies effective labour $e_i \ell_{i,t}$ and maximises $\sum_{t=0}^{\infty} \beta^t \left[\log c_{i,t} - \ell_{i,t}^2/2\right]$ subject to $\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t w_t e_i \ell_{i,t}$, with $\int_0^1 e_i \, di = 1$. There is no initial wealth.

1. From profit maximisation ($\max_L \{A_t L - w_t L\}$), derive the equilibrium wage w_t. What market structure is assumed?

2. Derive the household's optimality conditions for $c_{i,t}$ and $\ell_{i,t}$ from the AD Lagrangian. Show that the *static labour–leisure condition* is $\ell_{i,t} = \lambda_i p_t w_t e_i / \beta^t$.

3. Using the result from Problem 1 ($p_t = \beta^t g^{-t}$, which still holds here since goods-market clearing gives $Y_t = A_t$ as derived below), show that $\ell_{i,t} = 1$ for all i and t, and hence $c^*_{i,t} = e_i A_t$. Interpret the constancy of labour supply.

4. Compare the competitive equilibrium to the social planner's solution. Does the First Welfare Theorem apply? Verify directly that the planner's optimal allocation matches the equilibrium.

Problem 4 – Sequential equilibrium: equivalence with Arrow–Debreu

Consider the endowment economy from Problem 1. Assume now that agents trade a one-period bond with price q_t at date t (paying one unit at $t+1$). Define $p_t \equiv \prod_{s=0}^{t-1} q_s$ (with $p_0 = 1$), so the gross interest rate $1 + r_{t+1} = 1/q_t$. Agent i enters period t with $a_{i,t}$ units of the bond. Initial asset holdings satisfy $\int_0^1 a_{i,0}\, di = 0$.

1. Write the period-t budget constraint. By iterating from $t = 0$ to T, show that

$$\sum_{t=0}^{T} p_t c_{i,t} + p_T q_T a_{i,T+1} = a_{i,0} + \sum_{t=0}^{T} p_t y_{i,t}.$$

(*Hint*: use $p_{t+1} = p_t q_t$ to simplify.)

2. State the no-Ponzi-game (nPG) condition. Taking $T \to \infty$ and using nPG, derive the lifetime budget constraint (LBC) $\sum_{t=0}^{\infty} p_t c_{i,t} \le a_{i,0} + \sum_{t=0}^{\infty} p_t y_{i,t}$. Show it holds with equality at an optimum with u strictly increasing.

3. In the log-utility endowment economy with $Y_t = Y_0 g^t$, the sequential equilibrium Euler equation gives $q_t = \beta\, u'(c_{i,t+1})/u'(c_{i,t})$. Using the equilibrium allocation $c_{i,t} = e_i Y_t$, show that $q_t = \beta/g$ is constant for all t. Verify that the implied AD prices $p_t = (q_0)^t = (\beta/g)^t$ coincide with Problem 1.

Problem 5 – Neoclassical growth model: equilibrium and social optimality

There is a representative household with log utility, a representative competitive firm with Cobb-Douglas technology $F(K, L) = AK^{\alpha}L^{1-\alpha}$ ($0 < \alpha < 1$, $A > 0$), and exogenous labour supply $L = 1$. Capital depreciates at rate $\delta \in [0, 1]$. Let p_t denote the AD price of output at t.

1. Define the Arrow–Debreu competitive equilibrium for this economy (three conditions: household maximisation, firm maximisation, market clearing). From the firm's static problem, derive $r_t = \alpha A K_t^{\alpha-1}$ and $w_t = (1-\alpha) A K_t^{\alpha}$.

2. The firm also chooses its investment plan. Show that profit maximisation requires

$$p_t = p_{t+1}\big(r_{t+1} + 1 - \delta\big).$$

Interpret this as an arbitrage condition between the bond and the capital asset.

3. From the household's Euler equation and the goods-market clearing condition $C_t + K_{t+1} - (1-\delta)K_t = AK_t^{\alpha}$, derive the aggregate Euler equation

$$\frac{C_{t+1}}{C_t} = \beta\big(1 - \delta + r_{t+1}\big).$$

4. Show that the Euler equation above and the resource constraint are identical to the first-order conditions of the social planner's problem $\max \sum_{t=0}^{\infty} \beta^t \log C_t$ subject to $C_t + K_{t+1} = AK_t^{\alpha} + (1-\delta)K_t$. What property of the production function is key for this equivalence?

Problem 6 – Recursive steady-state equilibrium: endowment economy

Return to the log-utility endowment economy with constant endowments y_i (independent of t) and one-period bond price q. Agent i enters period t with a_i units of the bond. Consider the dynamic programme:

$$V_i(a) = \max_{a'} \big\{ \log(a + y_i - q a') + \beta V_i(a') \big\} \quad \forall a.$$

1. Define the recursive steady-state competitive equilibrium for this economy (you may quote directly from the text). Identify the three objects that must be determined: a bond price q, policy functions $g_i^*(a)$, and stationary asset holdings a_i^*.

2. Write the functional Euler equation. Evaluating it at the stationary point $a = a_i^*$ where $g_i^*(a_i^*) = a_i^*$, show that $q = \beta$.

3. Verify that the policy function $g_i^*(a) = a$ (hold assets fixed for all a, not just at a_i^*) satisfies the functional Euler equation for any a. Derive the value function $V_i^*(a) = \log(y_i + (1-\beta)a)/(1-\beta)$. What does this equilibrium imply for the distribution of wealth?

Problem 7 – Recursive competitive equilibrium: neoclassical growth model

Consider the neoclassical growth model from Problem 5 with log utility, $F(K, L) = AK^{\alpha}$ (i.e., $L = 1$), and a representative agent. In the recursive competitive equilibrium, the agent solves

$$V(k, K) = \max_{k'} \left\{ \log\big((1 - \delta + r(K))k + w(K) - k'\big) + \beta V(k', G(K)) \right\} \quad \forall\, (k, K),$$

where k is individual capital, K is aggregate capital, $r(K) = F_1(K, 1)$, $w(K) = F_2(K, 1)$, and $G(K)$ is the aggregate law of motion.

1. Explain why the state vector is (k, K) rather than just k. What additional object $G(K)$ is needed and why?

2. State the three conditions defining a recursive competitive equilibrium. (Individual optimality, firm optimality, and the consistency condition $G^*(K) = g^*(K, K)$.)

3. Derive the functional Euler equation for the individual policy $g(k, K)$. Evaluate it at $k = K$ and use the resource constraint $C = F(K, 1) - G(K) + (1 - \delta)K$ to show that the resulting functional equation for $G(K)$ coincides with the social planner's functional Euler equation.

4. Assume $\delta = 1$. Guess $G(K) = sAK^{\alpha}$ and $g(k, K) = sAK^{\alpha-1}k + \mu(K)$ (individual saving proportional to own capital plus a correction). Substitute into the consistency condition and the Euler equation to determine s. What does the linear form of g in k imply for aggregation?

Problem 8 – OLG model: endowment economy

Each period a new cohort is born and lives for two periods. Cohort t ($t \geq 0$) maximises $u(c_t^y) + \beta u(c_{t+1}^o)$ with $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, $\sigma = 2$, $\beta = 0.90$, subject to $c_t^y + q_t a_{t+1} = \omega_y$ and $c_{t+1}^o = \omega_o + a_{t+1}$. Endowments: $\omega_y = 1$, $\omega_o = 0.3$. The initial old consume $c_0^o = \omega_o$.

1. Define the sequential competitive equilibrium. Show that in equilibrium $a_{t+1}^* = 0$ for all t and that the equilibrium is autarky: $c_t^{y*} = \omega_y$, $c_t^{o*} = \omega_o$.

2. Evaluating the Euler equation at autarky, derive the bond price:

$$q_t = \beta \left(\frac{\omega_y}{\omega_o} \right)^{\sigma}$$

and compute its numerical value. Is the net real interest rate positive or negative? Contrast this with the dynastic log-utility endowment model with constant aggregate endowment.

3. Suppose the government mandates a *pay-as-you-go* (PAYG) pension: each young agent contributes $\tau \geq 0$ and each old agent receives τ. Let $W(\tau) = u(\omega_y - \tau) + \beta u(\omega_o + \tau)$ be the lifetime utility of any cohort born after the policy is implemented. Show that $W'(0) > 0$ with the given parameters, so a small PAYG transfer is a Pareto improvement. Compute the welfare-maximising transfer τ^*.

4. What does the existence of a welfare-improving PAYG transfer imply for Pareto efficiency of the laissez-faire equilibrium? Explain why the First Welfare Theorem fails in this OLG setting. (Hint: how many agents are involved at any given time?)

Problem 9 – OLG model: neoclassical growth

Two-period OLG economy. Only the young work ($e_y = 1$, $e_o = 0$). Production: $F(K, L) = K^\alpha L^{1-\alpha}$ with $\delta = 1$. Preferences: $\log c^y + \beta \log c^o$. Parameters: $\alpha = 0.36$, $\beta = 0.45$. At time t, the young earn wage $w_t = (1-\alpha)k_t^\alpha$ (per unit of labour, with aggregate $L = 1$) and invest k_{t+1} for old-age consumption $c^o_{t+1} = r_{t+1}k_{t+1}$ (where $r_{t+1} = \alpha k_{t+1}^{\alpha-1}$ and $\delta = 1$ so gross return equals the marginal product).

1. Write the young agent's problem. Derive the Euler equation and show that the capital accumulation equation is

$$k_{t+1} = \frac{\beta(1-\alpha)}{1+\beta}\, k_t^\alpha.$$

2. Find the unique positive steady state $\bar{k}$ analytically. Show that

$$\bar{k} = \left[\frac{\beta(1-\alpha)}{1+\beta}\right]^{1/(1-\alpha)}.$$

Compute the numerical value with $\alpha = 0.36$, $\beta = 0.45$.

3. Show that the gross steady-state return on capital is

$$\bar{R} = \frac{\alpha(1+\beta)}{\beta(1-\alpha)}.$$

Derive the condition on α and β under which $\bar{R} < 1$ (*dynamic inefficiency*). With $\beta = 0.45$, compute the threshold $\bar{\alpha}$ such that $\bar{R} = 1$. Is the economy with $\alpha = 0.36$ dynamically efficient?

4. The dynamics are governed by $k_{t+1} = h(k_t)$ where $h(k) = \frac{\beta(1-\alpha)}{1+\beta}k^\alpha$. Show that $h'(\bar{k}) = \alpha$ so the slope of the capital accumulation curve at the steady state equals the capital share. What does this imply for the speed of convergence? Compare with the dynastic model (slope $a_1 < 1$ from Problem 8 of the Chapter 4 problem set).

Problem 10 – MATLAB: transition dynamics and OLG vs. dynastic comparison

This problem asks you to solve the neoclassical growth model numerically and compare the dynastic and OLG economies.
Dynastic model: $F(K, 1) = K^\alpha$, $\delta = 0.10$, $\sigma = 2$ (CRRA utility), $\beta = 0.96$, $\alpha = 0.36$. **OLG model:** same $\alpha = 0.36$, $\beta = 0.45$, $\delta = 1$, log utility, $e_y = 1$, $e_o = 0$ (capital accumulation from Problem 9).

1. For the **dynastic model**, compute $\bar{k}$, $\bar{c}$, $\bar{r}$, and $\bar{w}$ analytically. Implement value function iteration (VFI) on a grid of 400 equally spaced points on $[0.3\bar{k}, 2\bar{k}]$ with sup-norm tolerance 10^{-8}. Report the number of iterations to convergence.

2. Using the VFI policy function, simulate 100 periods starting from $K_0 = 0.5\bar{k}$. Plot the paths of K_t, C_t, r_t, and w_t in a 2×2 figure. Mark the steady-state values with horizontal dashed lines.

3. For the **OLG model**, compute $\bar{k}$ and $\bar{R}$ using the formulas from Problem 9. Simulate the OLG capital path $k_{t+1} = \beta(1-\alpha)k_t^\alpha/(1+\beta)$ for 50 periods from $k_0 \in \{0.5\bar{k}, 2\bar{k}\}$. On a single figure, plot the cobweb diagram (the function $h(k)$ and the 45° line) and the time path of capital.

4. On a single figure, plot the *normalised* capital paths $K_t/\bar{k}^{\text{dyn}}$ (dynastic) and $k_t/\bar{k}^{\text{OLG}}$ (OLG), both starting from 0.5 of their respective steady states. Compare the speed of convergence to steady state. Report the number of periods for each model to come within 1% of its steady state. Relate the difference to the slopes a_1^{dyn} (VFI numerical slope at $\bar{k}$) and $h'(\bar{k}) = \alpha$ (OLG analytical slope).