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Econ 0107 – Macroeconomics I – Franck Portier

Problem Set: Welfare

Chapter 6 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook

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Problem 1 – The First Welfare Theorem in an Edgeworth box

Consider a static exchange economy with two goods, x and y, and two consumers, A and B. Consumer A has utility $a\log x_A + (1-a)\log y_A$ and endowment $(1,0)$; consumer B has utility $b\log x_B + (1-b)\log y_B$ and endowment $(0,1)$, with $a, b \in (0,1)$. Normalise the price of good x to one and let p denote the price of good y.

1. Compute the demand functions of both consumers and the competitive equilibrium price p^*. Show that the equilibrium allocation is $(x_A, y_A) = (a, b)$ and $(x_B, y_B) = (1-a, 1-b)$.

2. Verify that the marginal rates of substitution of A and B are equal at the equilibrium allocation. For $a = b = 1/2$, characterise the set of all Pareto-optimal allocations (the *contract curve*) and check that the equilibrium allocation lies on it.

3. From now on, let $a = b = 1/2$. The government levies a proportional tax $\tau > 0$ on consumer A's purchases of good y only (so A pays $(1+\tau)p$ per unit of y), and rebates the revenue to A as a lump-sum transfer $T = \tau p\, y_A$, which A takes as given. Compute the new equilibrium price and allocation. Show that

$$\mathrm{MRS}_A = \frac{2+\tau}{2(1+\tau)} \neq \mathrm{MRS}_B = \frac{2+\tau}{2}.$$

4. Let $\tau = 1$. Find all allocations on the diagonal of the box, $(x_A, y_A) = (z, z)$, $(x_B, y_B) = (1-z, 1-z)$, that make *both* consumers at least as well off as under the taxed equilibrium. Show that this set is a non-empty interval, so the taxed equilibrium is not Pareto optimal. Which step of the proof of the First Welfare Theorem breaks down?

Problem 2 – Negishi weights and the Pareto frontier

Two infinitely-lived agents $i \in \{1,2\}$ have preferences $\sum_{t=0}^{\infty} \beta^t u(c_{i,t})$, $\beta \in (0,1)$. Endowments alternate: agent 1 receives ω in even periods ($t = 0, 2, 4, \dots$) and 0 in odd periods; agent 2 receives ω in odd periods and 0 in even periods. The aggregate endowment is therefore constant and equal to ω. Markets are complete (Arrow–Debreu) and $p_0 = 1$.

1. A planner maximises $\mu_1 U_1 + \mu_2 U_2$ with $\mu_i > 0$ subject to $c_{1,t} + c_{2,t} \leq \omega$ for all t. For CRRA utility $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, show that each agent receives a constant share of the aggregate endowment,

$$c_{1,t} = \frac{\mu_1^{1/\sigma}}{\mu_1^{1/\sigma} + \mu_2^{1/\sigma}}\,\omega \qquad \forall t,$$

and explain why the solution is Pareto optimal for any (μ_1, μ_2).

2. Let $u = \log$. Writing $s \in (0,1)$ for agent 1's consumption share, express (U_1, U_2) as functions of s and describe the Pareto frontier in the (U_1, U_2) plane. Is it concave? What weights (μ_1, μ_2) select a given point s on the frontier?

3. Still with $u = \log$, compute the competitive equilibrium: AD prices p_t, the Lagrange multipliers λ_i and consumption $c_{i,t}$. Show that the equilibrium coincides with the planner's solution for $\mu_i = 1/\lambda_i$ and compute the implied ratio μ_1/μ_2. Why does agent 1 receive a larger weight?

4. A utilitarian government wants to implement the equal-weight optimum $\mu_1 = \mu_2$. By the Second Welfare Theorem, this can be achieved as a competitive equilibrium with a lump-sum transfer at date 0. Compute the present-value transfer T from agent 1 to agent 2 that implements $c_{1,t} = c_{2,t} = \omega/2$. Why is it essential that the transfer be lump-sum?

Problem 3 – Distortionary labour taxation and the consumption-equivalent welfare loss

A static economy has a representative household with utility

$$u(c,\ell) = \log c - \frac{\ell^{1+1/\theta}}{1+1/\theta}, \qquad \theta > 0,$$

and a competitive firm with technology $y = A\ell$. Labour income is taxed at rate $\tau \in [0,1)$ and the revenue is rebated lump-sum, $T = \tau w \ell$.

1. Derive the equilibrium wage, and show that equilibrium hours satisfy $\ell_d = (1-\tau)^{\theta/(1+\theta)}$ and consumption $c_d = A\ell_d$. Show that the allocation with $\tau = 0$ (or equivalently with any lump-sum tax) coincides with the planner's optimum $(c_l, \ell_l) = (A, 1)$.

2. Define the consumption-equivalent welfare loss Δ by $u\big(c_l(1-\Delta), \ell_l\big) = u(c_d, \ell_d)$. Show that

$$1 - \Delta = \exp\Big\{\eta\big[\log(1-\tau) + \tau\big]\Big\}, \qquad \eta \equiv \frac{\theta}{1+\theta},$$

and that Δ does not depend on A.

3. Show that for small τ, $\Delta \approx \eta\tau^2/2$. Interpret this "Harberger triangle" result: why is the welfare cost second order in τ and increasing in the Frisch elasticity θ? What happens when $\theta \to 0$?

4. Compute Δ exactly and with the approximation for $\theta = 1$, $\tau = 0.3$. Compute also the alternative measure Δ' defined by $u\big(c_d(1+\Delta'), \ell_d\big) = u(c_l, \ell_l)$ and comment on the difference.

Problem 4 – Capital income taxation in the neoclassical growth model

A representative household maximises $\sum_{t=0}^{\infty} \beta^t \log c_t$ and supplies one unit of labour inelastically. A competitive firm produces $y_t = k_t^{\alpha}$ and capital depreciates at rate δ. The government taxes *net* capital income $(r_t - \delta)k_t$ at rate τ_k and rebates the revenue lump-sum, $T_t = \tau_k(r_t - \delta)k_t$. The household's budget is

$$c_t + k_{t+1} = k_t + (1-\tau_k)(r_t - \delta)k_t + w_t + T_t.$$

Parameters: $\alpha = 1/3$, $\beta = 0.96$, $\delta = 0.10$.

1. Derive the household's Euler equation and compare it with the planner's Euler equation. Identify the wedge between the marginal rate of substitution and the marginal rate of transformation between c_t and c_{t+1}.

2. Derive the steady-state capital stock $k_{ss}(\tau_k)$. Compute it for $\tau_k = 0$ and $\tau_k = 0.3$, as well as the golden-rule capital stock k_{GR}.

3. Compute steady-state consumption in both cases. Is steady-state consumption higher or lower with the tax? Would your answer change if $k_{ss}(0)$ were above k_{GR}? Why is it impossible in this model?

4. A student argues: "to evaluate the welfare cost of τ_k, compare $\log c_{ss}(0.3)$ with $\log c_{ss}(0)$, divided by $1-\beta$." Explain why this is not the right welfare comparison. How would you correctly compute the consumption-equivalent loss of introducing the tax in an economy that starts at $k_{ss}(0)$? (No computation needed.)

Problem 5 – Production externalities and the Pigouvian tax

A static economy has a unit mass of identical competitive firms $j \in [0,1]$ producing $y_j = A(\bar{y})\, k^{\alpha} \ell_j^{1-\alpha}$, where capital k is fixed and $\bar{y} = \int_0^1 y_j \, dj$ is average output, taken as given by each firm. The externality is $A(\bar{y}) = \bar{y}^{-\gamma}$ with $\gamma > -1$ (congestion if $\gamma > 0$, spillover if $\gamma < 0$). The representative household owns capital and the firms and has utility

$$\log c - B\,\frac{\ell^{1+1/\theta}}{1+1/\theta}.$$

1. Derive the wage and the household's labour supply condition. Show that equilibrium hours satisfy $1-\alpha = B\ell^{1+1/\theta}$, independently of γ.

2. The planner takes into account that $\bar{y} = y$. Show that in a symmetric allocation $\log y = \frac{1}{1+\gamma}\big[\alpha \log k + (1-\alpha)\log \ell\big]$, and that efficient hours satisfy

$$\frac{1-\alpha}{1+\gamma} = B\ell^{1+1/\theta}.$$

Check that this is the slides' condition $(1-\alpha)/\big(1 - A'(y)k^{\alpha}\ell^{1-\alpha}\big) = B\ell^{1+1/\theta}$. Do households work too much or too little when $\gamma > 0$? When $\gamma < 0$?

3. The government taxes firms' revenue at rate τ (firms receive $(1-\tau)$ per unit sold) and rebates the proceeds lump-sum to households. Find the tax rate τ^* that restores efficiency. Show that τ^* equals the marginal external damage of one unit of output, evaluated at the efficient allocation, as a fraction of the private marginal value of output.

4. Let $\alpha = 1/3$, $B = 1$, $\theta = 1$ and $\gamma = 0.2$. Compute equilibrium hours, efficient hours and τ^*. Why does the proof of the First Welfare Theorem fail in this economy?

Problem 6 – Missing markets: borrowing constraints

Two infinitely-lived agents have preferences $\sum_{t=0}^{\infty} \beta^t \log c_{i,t}$. Endowments alternate: agent 1 receives $\omega/3$ in even periods ($t = 0, 2, \dots$) and $2\omega/3$ in odd periods; agent 2 receives the reverse. Agents trade a one-period bond at price q_t and start with zero assets.

1. Without borrowing constraints, compute the competitive equilibrium: bond prices q_t, and constant consumption levels c_1 and c_2 as fractions of ω. Which agent consumes more, and why?

2. Now impose $a_{i,t+1} \geq 0$ for both agents. Write agent i's Euler condition with the constraint, and explain why it holds as an inequality. Show that the unique equilibrium is autarky.

3. Show that the equilibrium bond price is $q_t = 2\beta$ for all t. Check that the agent who would like to borrow indeed satisfies her (inequality) Euler condition at that price. Why is the interest rate set by the unconstrained agent? For which values of β is the net interest rate negative?

4. Let $\beta = 0.9$. For each agent, compute the consumption-equivalent loss Δ_i from moving from the unconstrained equilibrium to autarky, defined by $\sum_t \beta^t \log\big(c_i(1-\Delta_i)\big) = U_i^{\text{aut}}$. Is autarky Pareto optimal?

Problem 7 – Monopolistic competition and CES aggregation

A representative household has preferences

$$\log C - \frac{L^{1+1/\varphi}}{1+1/\varphi}, \qquad C = \left(\int_0^1 c(i)^{\frac{\varepsilon-1}{\varepsilon}}\, di\right)^{\frac{\varepsilon}{\varepsilon-1}}, \qquad \varepsilon > 1,\ \varphi > 0.$$

Each variety i is produced by a monopolist with technology $c(i) = A\ell(i)$. The household owns all firms and receives their profits Π.

1. For a given level of spending $Y = \int_0^1 p(i)c(i)\, di$, show that the cost-minimising demand for variety i is $c(i) = C\,(p(i)/P)^{-\varepsilon}$ with $P = \big(\int_0^1 p(i)^{1-\varepsilon} di\big)^{1/(1-\varepsilon)}$, and that $Y = PC$.

2. Solve the firm's problem taking C and P as given, and show that the optimal price is a constant markup over marginal cost, $p(i) = \frac{\varepsilon}{\varepsilon-1}\frac{w}{A}$. Normalising $P = 1$, derive the equilibrium real wage and the share of profits in output.

3. Show that equilibrium hours satisfy $L^{1+1/\varphi} = 1 - 1/\varepsilon$, whereas the planner chooses $L = 1$. Show that a revenue subsidy s on sales, financed by a lump-sum tax, restores efficiency if $1 + s = \varepsilon/(\varepsilon - 1)$.

4. Let $\varepsilon = 6$ and $\varphi = 1$. Compute the markup, equilibrium hours and the consumption-equivalent welfare loss Δ relative to the efficient allocation, defined as in Problem 3. Why does the proof of the First Welfare Theorem fail here, even though there are no taxes, externalities or missing markets?

Problem 8 – OLG endowment economy: Balasko–Shell and Pareto optimality

Each period a cohort of $N_t = (1+n)^t$ identical agents is born, with $n \geq 0$. Agents live two periods and have utility $\log c_y + \log c_o$, with endowments (ω_y, ω_o). The initial old (cohort -1) consume ω_o at $t = 0$ in autarky. Agents can trade a one-period bond at price q_t among themselves.

1. Let $n = 0$. Show that the competitive equilibrium is autarky with $q_t = \omega_y/\omega_o$. Compute the Arrow–Debreu prices p_t and use the Balasko–Shell criterion to determine for which endowments the equilibrium is Pareto optimal.

2. Still with $n = 0$, a government runs a stationary pay-as-you-go (PAYG) scheme: each young agent pays τ, each old agent receives τ. Let $W(\tau) = \log(\omega_y - \tau) + \log(\omega_o + \tau)$. Show that a small $\tau > 0$ is a Pareto improvement if and only if $\omega_y > \omega_o$, and compute the optimal stationary τ^*.

3. Now let $n > 0$. With the same PAYG scheme, each old agent receives $(1 + n)\tau$. Show that a small PAYG scheme is a Pareto improvement if and only if $\omega_o/\omega_y < 1 + n$. Interpret this condition by comparing the return on the bond in autarky with the "return" on the PAYG scheme.

4. Let $n = 0$, $\omega_y = 3$, $\omega_o = 1$ and consider the stationary allocation $c_y = c_o = 2$. We want to show that no allocation Pareto dominates it. Suppose an alternative gives the initial old $2 + \varepsilon_0$ with $\varepsilon_0 > 0$, so cohort 0 consumes $2 - \varepsilon_0$ when young. Show that for cohort t not to be worse off, it must receive at least $2 + \varepsilon_{t+1}$ when old, where

$$\varepsilon_{t+1} = \frac{2\varepsilon_t}{2 - \varepsilon_t}.$$

Show that $1/\varepsilon_t = 1/\varepsilon_0 - t/2$ and conclude. Why does the same argument *not* apply to the autarky allocation $(3, 1)$?

Problem 9 – Dynamic inefficiency in the Diamond OLG model

Consider the two-period OLG model of Problem 9 in the Chapter 5 problem set: constant population, only the young work, $F(k, 1) = k^\alpha$, $\delta = 1$, preferences $\log c_y + \beta \log c_o$. Recall that

$$k_{t+1} = \frac{\beta(1-\alpha)}{1+\beta} k_t^\alpha, \qquad \bar{k} = \left[\frac{\beta(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}}, \qquad \bar{R} = \frac{\alpha(1+\beta)}{\beta(1-\alpha)}.$$

Parameters: $\alpha = 0.2$, $\beta = 0.9$.

1. Compute $\bar{k}$, $\bar{R}$ and the threshold $\bar{\alpha} = \beta/(1 + 2\beta)$. Is the steady state dynamically efficient? Compute steady-state consumption of the young $\bar{c}_y = (1-\alpha)\bar{k}^\alpha - \bar{k}$, of the old $\bar{c}_o = \alpha \bar{k}^\alpha$, and aggregate consumption $\bar{C}$.

2. The golden-rule capital stock maximises steady-state aggregate consumption $C = k^\alpha - k$. Derive k_{GR} and show that $\bar{k} > k_{GR}$ if and only if $\bar{R} < 1$.

3. Starting at $k_0 = \bar{k}$, a planner decides to invest $k_{t+1} = k_{GR}$ for all $t \geq 0$. Compute aggregate consumption C_0 at $t = 0$ and C_t for $t \geq 1$, and compare them with $\bar{C}$. Explain how the planner can split C_t between young and old so that *every* generation, including the initial old, is strictly better off than in the laissez-faire equilibrium.

4. Explain intuitively why the market economy over-accumulates capital, and why the same cannot happen in the dynastic neoclassical growth model. Relate your answer to the Balasko–Shell criterion and the theorem on dynamic efficiency on the slides.

Problem 10 – MATLAB: quantifying welfare losses and dynamic inefficiency

This problem uses the results of Problems 3, 7, 8 and 9.

1. Labour tax. Using the exact formula of Problem 3, plot the consumption-equivalent loss $\Delta(\tau)$ for $\tau \in [0, 0.6]$ and $\theta \in \{0.5, 1, 2\}$, together with the Harberger approximation $\eta\tau^2/2$. Report $\Delta(0.3)$ for each θ. On a second panel, plot tax revenue $R(\tau) = \tau A \ell_d(\tau)$ (with $A = 1$) and the ratio $\Delta(\tau)/R(\tau)$. Comment.

2. Monopolistic competition. Using Problem 7, plot equilibrium hours $L(\varepsilon)$ and the welfare loss $\Delta(\varepsilon)$ for $\varepsilon \in [1.5, 20]$, with $\varphi = 1$. Report Δ for $\varepsilon \in \{2, 6, 11, 20\}$. Comment on the size of these losses and on the role of ε.

3. OLG perturbations. For the $(2, 2)$ allocation of Problem 8, iterate $\varepsilon_{t+1} = 2\varepsilon_t/(2 - \varepsilon_t)$ starting from $\varepsilon_0 \in \{0.1, 0.05, 0.01\}$ and report the first date at which $\varepsilon_t \geq 2$. Plot ε_t against t (log scale on the vertical axis). Compare with the closed-form solution.

4. Dynamic efficiency map. For the Diamond model of Problem 9, compute $\bar{R}(\alpha, \beta)$ on a grid $\alpha \in [0.05, 0.6]$, $\beta \in [0.1, 1]$ and plot the contour $\bar{R} = 1$, shading the dynamically inefficient region. Add the analytical boundary $\alpha = \beta/(1 + 2\beta)$. Then, for $\alpha = 0.2$, $\beta = 0.9$, simulate 10 periods of the laissez-faire economy and of the golden-rule policy of Problem 9 Q3 starting from $k_0 = \bar{k}$, and plot aggregate consumption C_t under both.