University College London, 2026–2027
Econ 0107 – Macroeconomics I – Franck Portier

Problem Set: Uncertainty

Chapter 7 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook

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Problem 1 – Markov chains

A worker is either employed (state 1) or unemployed (state 2). Each month an employed worker loses her job with probability s and an unemployed worker finds a job with probability f, with $s, f \in (0,1)$:

$$P = \begin{pmatrix} 1-s & s \\ f & 1-f \end{pmatrix}.$$

A continuum of workers face independent draws, so that the unemployment rate u_t equals the probability of being unemployed.

1. Show that $u_{t+1} = s + (1-s-f)\,u_t$. Derive the stationary unemployment rate $\bar{u}$ in two ways: (i) as a fixed point of this equation, and (ii) as the eigenvector of P' associated with the unit eigenvalue.

2. Show that $u_t - \bar{u} = \lambda^t (u_0 - \bar{u})$ with $\lambda = 1 - s - f$, and check that λ is the second eigenvalue of P. With $s = 0.02$ and $f = 0.30$, compute $\bar{u}$ and the half-life of a deviation of unemployment from $\bar{u}$.

3. Compute the stationary distribution of the three-state chain

$$P_3 = \begin{pmatrix} 0.90 & 0.10 & 0 \\ 0.05 & 0.90 & 0.05 \\ 0 & 0.20 & 0.80 \end{pmatrix}.$$

Is the chain irreducible? Aperiodic? What does this imply for $\lim_{t\to\infty} \pi_0 P_3^t$?

4. For each of the following transition matrices, find all stationary distributions and say whether $\pi_0 P^t$ converges for every π_0:

$$P_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_c = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Relate your answers to the irreducibility and aperiodicity conditions. Is irreducibility necessary for uniqueness?

Problem 2 – Autoregressive processes and linear stochastic difference equations

Let ε_t satisfy $\mathbb{E}_{t-1}[\varepsilon_t] = 0$, $\mathbb{E}_{t-1}[\varepsilon_t^2] = 1$ and $\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_{t+s}] = 0$ for $s > 0$.

1. Consider the AR(1) process $x_t = \rho x_{t-1} + b\varepsilon_t + (1-\rho)\mu$ with $|\rho| < 1$. Derive its moving-average representation and use it to show that $\mathbb{E}[x_t] = \mu$, $\mathrm{Var}(x_t) = b^2/(1-\rho^2)$ and $\mathrm{Corr}(x_t, x_{t+j}) = \rho^j$. Use the law of iterated expectations to compute $\mathbb{E}_t[x_{t+h}]$ and interpret ρ.

2. Now consider the AR(2) process $y_t = 1.2\,y_{t-1} - 0.35\,y_{t-2} + \varepsilon_t$. Write it as a first-order vector system $x_t = A x_{t-1} + B\varepsilon_t$ with $x_t = (y_t, y_{t-1})'$. Compute the eigenvalues of A and show that the process is stationary. What is its unconditional mean?

3. Let $\Gamma(0) \equiv \mathbb{E}\big[(x_t - \mu)(x_t - \mu)'\big]$ denote the unconditional covariance matrix of x_t. Show that it satisfies the Lyapunov equation $\Gamma(0) = A\Gamma(0)A' + BB'$, and use it to compute $\mathrm{Var}(y_t)$ and the first-order autocorrelation of y_t. (*Hint*: by stationarity, $\Gamma(0) = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}$, where $\gamma_0 = \mathrm{Var}(y_t)$ and $\gamma_1 = \mathrm{Cov}(y_t, y_{t-1})$.)

4. Compute the impulse response $F(h) = e_1' A^h B$ of y to a unit shock for $h = 0, \ldots, 4$, where $e_1 = (1,0)'$. Show that $F(h) = \big(0.7^{h+1} - 0.5^{h+1}\big)/0.2$. Describe its shape and contrast it with the impulse response of an AR(1).

Problem 3 – Expected utility and risk aversion

1. Compute the coefficients of absolute risk aversion $-u''(c)/u'(c)$ and of relative risk aversion $-cu''(c)/u'(c)$ for (i) CARA utility $u(c) = -\exp(-\alpha c)$; (ii) CRRA utility $u(c) = (c^{1-\sigma}-1)/(1-\sigma)$; (iii) quadratic utility $u(c) = c - \frac{\gamma}{2}c^2$ for $c < 1/\gamma$. Why is quadratic utility an unattractive description of attitudes towards risk?

2. Show that if $v(c) = a\,u(c) + b$ with $a > 0$, then v and u rank all lotteries identically and have the same coefficient of absolute risk aversion. Why are the coefficients of risk aversion normalised by $u'(c)$?

3. A consumer faces the lottery $c \in \{50, 150\}$ with equal probabilities. The *certainty equivalent* c^{CE} satisfies $u(c^{\mathrm{CE}}) = \mathbb{E}[u(c)]$ and the *risk premium* is $\mathbb{E}[c] - c^{\mathrm{CE}}$. Compute both for CRRA utility with $\sigma = 1$ (log) and $\sigma = 2$, and for CARA utility with $\alpha = 0.02$.

4. Show that for small risks, the risk premium is approximately $\frac{1}{2}\,\mathcal{A}(\bar{c})\,\mathrm{Var}(c)$, where $\mathcal{A}$ is the coefficient of absolute risk aversion and $\bar{c} = \mathbb{E}[c]$ (the Arrow–Pratt approximation). Compare the approximation with the exact values of Q3. Explain why, with CRRA utility, the same parameter σ governs both risk aversion and the elasticity of intertemporal substitution.

Problem 4 – Portfolio choice: CRRA versus CARA

An investor with wealth $W > 0$ invests A in a risky asset with gross return Z and $W - A$ in a risk-free asset with gross return R_f, and consumes $c = R_f(W - A) + ZA$ at date 1. The risky return takes two equally likely values, $Z \in \{Z_h, Z_l\}$, with $Z_h = 1.30$, $Z_l = 0.90$, and $R_f = 1.02$. Let $x_h = Z_h - R_f$ and $x_l = Z_l - R_f$ denote the excess returns.

1. Write the first-order condition for A. Show that a risk-neutral investor would choose a corner solution, and that any risk-averse investor holds a strictly positive amount of the risky asset when $\mathbb{E}[Z] > R_f$. (*Hint*: evaluate the derivative of expected utility at $A = 0$.)

2. Let $u(c) = c^{1-\sigma}/(1-\sigma)$. Show that the risky share $\theta = A/W$ does not depend on W and that

$$\theta = \frac{R_f(\kappa - 1)}{x_h - \kappa x_l}, \qquad \kappa \equiv \Big(\frac{x_h}{-x_l}\Big)^{1/\sigma}.$$

Compute θ for $\sigma = 2$ and $\sigma = 5$ and interpret.

3. Let $u(c) = -\exp(-\alpha c)$. Show that the optimal amount A does not depend on W and that $A = \log(x_h/(-x_l))/\big(\alpha(x_h - x_l)\big)$. Compute A for $\alpha = 2$ and the implied risky share for $W = 2$ and $W = 10$.

4. In the data, richer households hold a larger share of their wealth in risky assets. Which of the two models is closer to the data? Using the risk-aversion coefficients of Problem 3, explain what feature of preferences would be needed to match the fact.

Problem 5 – The two-period stochastic growth model and precautionary saving

At date 0 the economy has capital K_0 and produces K_0^{α}. At date 1, TFP $A_1(\omega_1)$ is random with probabilities $\pi_1(\omega_1)$. The planner maximises $u(C_0) + \beta \sum_{\omega_1} \pi_1(\omega_1) u\big(C_1(\omega_1)\big)$ subject to

$$K_1 + C_0 = K_0^{\alpha} + (1-\delta)K_0, \qquad C_1(\omega_1) = A_1(\omega_1)K_1^{\alpha} + (1-\delta)K_1 \quad \forall \omega_1.$$

1. Explain why the planner chooses a *contingent plan*. If the economy lasted $T + 1$ periods and $\omega_t \in \{0, 1\}$ at each date $t \geq 1$ (with ω_0 known), how many distinct consumption levels would a contingent plan specify?

2. Set up the Lagrangian with multipliers λ_0 and $\lambda_1(\omega_1)$ and derive the stochastic Euler equation

$$u'(C_0) = \beta \sum_{\omega_1} \pi_1(\omega_1)\, u'\big(C_1(\omega_1)\big)\big(\alpha A_1(\omega_1) K_1^{\alpha-1} + 1 - \delta\big).$$

Why is the return to saving weighted by marginal utility rather than simply averaged?

3. Let $u = \log$ and $\delta = 1$. Show that $K_1 = \frac{\alpha\beta}{1+\alpha\beta} K_0^{\alpha}$, whatever the distribution of A_1. Interpret.

4. Let $\delta = 1$ and $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ with $\sigma \neq 1$. Show that K_1 solves

$$(K_0^{\alpha} - K_1)^{-\sigma} = \alpha\beta\, K_1^{\alpha(1-\sigma)-1}\, \mathbb{E}\big[A_1^{1-\sigma}\big].$$

Suppose $A_1 \in \{1-\Delta, 1+\Delta\}$ with equal probabilities. Show that a mean-preserving increase in risk ($\Delta \uparrow$) raises saving K_1 if $\sigma > 1$ and lowers it if $\sigma < 1$. Relate this to the competing income and substitution effects of risk on the return to saving.

Problem 6 – The recursive stochastic growth model: an exact solution

TFP follows a first-order Markov process with transition probabilities $\pi(A'|A)$, and $f(A,K) = AK^{\alpha} + (1-\delta)K$. The planner solves

$$V(A,K) = \max_{K'} \Big\{ u\big(f(A,K) - K'\big) + \beta \sum_{A'} \pi(A'|A) V(A',K') \Big\}.$$

1. Derive the first-order and envelope conditions and show that the policy function $K' = g(A,K)$ satisfies the functional Euler equation

$$u'\big(f(A,K) - g(A,K)\big) = \beta \sum_{A'} \pi(A'|A)\, u'\Big(f\big(A', g(A,K)\big) - g\big(A', g(A,K)\big)\Big) f_2\big(A', g(A,K)\big).$$

Explain, using the law of iterated expectations, why a one-period-ahead expectation in the Bellman equation is enough.

2. Let $u = \log$ and $\delta = 1$ (Brock–Mirman). Guess $g(A,K) = sAK^{\alpha}$ and verify that the functional Euler equation holds with $s = \alpha\beta$, for *any* Markov process for A.

3. Let $\log A_t = \rho \log A_{t-1} + \sigma_\varepsilon \varepsilon_t$. Show that $x_t \equiv (\log A_t, \log K_{t+1})'$ follows a linear stochastic difference equation $x_t = \mathcal{A} x_{t-1} + \mathcal{B}\varepsilon_t + \mathcal{C}$ and give $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. Compute the unconditional mean of $\log K$.

4. For $\rho = 0$, show that $\mathrm{Var}(\log K) = \sigma_\varepsilon^2/(1-\alpha^2)$. For general ρ, show that the impulse response of $\log K_{t+1+h}$ to a unit innovation ε_t is $\sigma_\varepsilon(\rho^{h+1} - \alpha^{h+1})/(\rho - \alpha)$, and that the response of $\log Y_{t+h}$ is $\sigma_\varepsilon\big[\rho^h + \alpha(\rho^h - \alpha^h)/(\rho-\alpha)\big]$. With $\alpha = 0.3$, $\beta = 0.96$, $\rho = 0.9$, $\sigma_\varepsilon = 0.01$, one can show that $\mathrm{Var}(\log K) = \sigma_\varepsilon^2(1+\alpha\rho)/[(1-\alpha^2)(1-\rho^2)(1-\alpha\rho)]$; compute the standard deviation of $\log K$ and compare it with that of $\log A$.

Problem 7 – Linearisation and certainty equivalence

TFP follows $A' = \rho A + (1-\rho)\bar{A} + \varepsilon'$ with $\mathbb{E}[\varepsilon'] = 0$. Linearising the functional Euler equation of Problem 6 around the deterministic steady state $(\bar{A}, \bar{K})$ gives $K' = \bar{K} + g_K(K - \bar{K}) + g_A(A - \bar{A})$, where g_K solves

$$g_K^2 - \big(1 + \beta^{-1} + \Theta\big) g_K + \beta^{-1} = 0, \qquad \Theta \equiv \frac{u'}{u''}\frac{f_{KK}}{f_K},$$

and

$$g_A = \frac{(1-\rho) f_A - \rho\, \frac{u'}{u''}\frac{f_{KA}}{f_K}}{f_K - g_K + 1 - \rho + \Theta},$$

with all derivatives evaluated at the steady state, where $\beta f_K = 1$.

1. Show that $\Theta > 0$. Show that the quadratic has two positive real roots whose product is β^{-1}, and that it is negative at $g_K = 1$. Conclude that one root lies in $(0,1)$ and the other above 1. Why must we select the smaller root?

2. Explain the principle of *certainty equivalence*: why does $\mathrm{Var}(\varepsilon')$ not appear in g_K or g_A? Contrast with Problem 5 Q4. Is certainty equivalence a property of the model or of the approximation?

3. Check the formulas on the Brock–Mirman case of Problem 6 ($u = \log$, $f = AK^{\alpha}$, $\bar{A} = 1$): show that they deliver $g_K = \alpha$ and $g_A = \bar{K}$, the derivatives of the exact policy $\alpha\beta AK^{\alpha}$ at the steady state.

4. Use the slides' calibration: $f(A,K) = AK^{\alpha} + (1-\delta)K$, $\alpha = 0.3$, $\delta = 0.02$, $\rho = 0.95$, $\bar{A} = 1$, $u = \log$, $\beta = 0.99$. Compute $\bar{K}$, $\bar{C}$, Θ, g_K and g_A. What is the half-life of a capital deviation? By how much (in %) does next period's capital rise after a 1% increase in TFP?

Problem 8 – Complete markets: insurance and risk sharing

Households $i \in \mathcal{I}$ have preferences $\sum_t \sum_{\omega^t} \beta^t \pi_t(\omega^t) u(c_{i,t}(\omega^t))$, receive stochastic endowments $y_{i,t}(\omega^t)$ of a perishable good, and trade at date 0 a complete set of Arrow–Debreu claims with prices $p_t(\omega^t)$.

1. Write household i's budget constraint and first-order conditions. Show that if prices are *actuarially fair*, $p_t(\omega^t) = \bar{p}_t \pi_t(\omega^t)$, consumption does not depend on ω^t. When can prices be actuarially fair in equilibrium?

2. Show that $u'(c_{i,t}(\omega^t))/u'(c_{j,t}(\omega^t)) = \lambda_i/\lambda_j$ for all t and ω^t. With CRRA utility, show that each household consumes a constant share of aggregate endowment $Y_t(\omega^t)$. What does this imply for the response of $c_{i,t}$ to $y_{i,t}$, holding Y_t fixed?

3. *Idiosyncratic risk only.* A unit continuum of ex-ante identical households are each employed with probability π_e (income y_e) or unemployed (income $y_u < y_e$), independently across households and over time, and trade takes place before any uncertainty is realised. Assuming a law of large numbers, what is aggregate income? Characterise the equilibrium consumption allocation and AD prices.

4. *Aggregate risk.* Two households have $u = \log$, $\beta = 0.9$. Each period an i.i.d. state $s \in \{1,2\}$ is drawn with probability $1/2$; endowments are $y_1(1) = 3$, $y_1(2) = 1$, $y_2(1) = y_2(2) = 1$, so that $Y(1) = 4$, $Y(2) = 2$. The initial state is $s_0 = 1$ and $p_0 = 1$. Compute the AD prices, household 1's consumption share, and the one-period Arrow prices $q(s'|s)$ under sequential trading. Compute the gross risk-free rate in each state. Are prices actuarially fair?

Problem 9 – Sequential trading, spanning and complete markets

1. Under sequential trading, household i faces $c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega_{t+1}) = y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$. Derive the Euler equation for each Arrow security. Show that the AD prices implied by the sequential equilibrium are $p_t(\omega^t) = \prod_{\tau=0}^{t-1} q_\tau(\omega_{\tau+1}|\omega^\tau)$. Explain the role of the natural borrowing limit (nPg) in establishing the equivalence with the AD equilibrium.

2. There are $S = 3$ states next period and two assets: a risk-free bond with payoff $(1,1,1)'$ and a stock with payoff $(1,2,3)'$. Write the payoff matrix D and its rank. Are markets complete? Give an example of a payoff vector that cannot be attained.

3. A call option on the stock with strike price 2 is introduced. What is its payoff vector? Show that markets are now complete, and find the portfolios θ_j^A (in bonds, stocks and options) that replicate the three Arrow securities.

4. The bond, stock and option sell for 0.95, 1.90 and 0.30. Compute the implied Arrow (state) prices, the risk-free rate and the risk-neutral probabilities. Now suppose the option sold for 0.50 instead. Show that there is an arbitrage opportunity, and construct it.

Problem 10 – Competitive equilibrium in the stochastic growth model

A representative household owns k_0 and one unit of labour supplied inelastically, and rents both to a competitive firm producing $y_t(\omega^t) = A_t(\omega^t) F(k_t(\omega^{t-1}), 1)$ with F CRS. Its budget is $c_t(\omega^t) + k_{t+1}(\omega^t) = \big(r_t(\omega^t) + 1 - \delta\big) k_t(\omega^{t-1}) + w_t(\omega^t)$.

1. State the firm's first-order conditions and derive the household's stochastic Euler equation. Write the definition of a sequential competitive equilibrium.

2. Show that goods-market clearing is implied by the household budget and the firm's conditions (Walras' law). Then show that the equilibrium allocation coincides with the planner's solution of Problem 6. Why do the welfare theorems continue to hold under uncertainty?

3. Suppose the household could also trade one-period Arrow securities contingent on ω_{t+1}. Explain why this would not change the equilibrium allocation. Use the Arrow-security Euler equation to price a one-period risk-free bond, q_t^f.

4. Let $m_{t+1} \equiv \beta u'(c_{t+1})/u'(c_t)$ and $R_{t+1}^k \equiv r_{t+1} + 1 - \delta$. Using the Euler equations for capital and for the bond, show that

$$\mathbb{E}_t[R_{t+1}^k] - R_{t+1}^f = -R_{t+1}^f \operatorname{Cov}_t\big(m_{t+1}, R_{t+1}^k\big), \qquad R_{t+1}^f = 1/q_t^f .$$

Is the expected return on capital above or below the risk-free rate in this model? Why?

Problem 11 – Incomplete markets and precautionary saving

A consumer maximises $\mathbb{E}_0 \sum_t \beta^t u(c_t)$, receives a stochastic income $y_t \in [y_{\min}, y_{\max}]$, $y_{\min} > 0$, following a Markov process, and can only trade a risk-free asset: $q a_{t+1} + c_t = a_t + y_t$, with $q = 1/(1+r)$ and $a_{t+1} \geq \underline{a}$.

1. Derive the *natural borrowing limit*, the largest debt the consumer could repay for sure. Show that if $u'(0) = \infty$, a consumer facing only this limit never chooses to borrow up to it.

2. Write the Bellman equation and derive the Euler condition when the borrowing constraint may bind: $u'(c) \geq \beta(1+r)\,\mathbb{E}[u'(c')|y]$, with equality if $a' > \underline{a}$. Why is this an incomplete-markets economy, and why does the consumer not choose a perfectly smooth consumption path?

3. *Two periods, CARA utility.* Let $u(c) = -\exp(-\alpha c)$, $a_0 = 0$, $\beta(1+r) = 1$, and let date-1 income be $y_1 \sim \mathcal{N}(\mu, \sigma_y^2)$; the borrowing constraint does not bind. With $R \equiv 1 + r$, show that

$$c_0 = \frac{R y_0 + \mu}{1+R} - \frac{\alpha \sigma_y^2}{2(1+R)}.$$

Identify the precautionary saving term and compute it for $\alpha = 2$, $\sigma_y = 0.2$, $r = 0.03$. (*Hint*: if $X \sim \mathcal{N}(m, v)$, $\mathbb{E}[e^X] = e^{m + v/2}$.)

4. Show that with quadratic utility there is no precautionary saving. Which property of u generates precautionary saving? Relate your answer to certainty equivalence (Problem 7). Finally, explain why we need $\beta(1+r) < 1$ for the consumer's assets to remain bounded, and describe the ergodic set of assets in the infinite-horizon problem.

Problem 12 – MATLAB: Markov chains, stochastic growth and self-insurance

This problem uses the results of Problems 1, 6, 7 and 11.

1. Markov chain. For the unemployment chain of Problem 1 ($s = 0.02$, $f = 0.30$), compute the stationary distribution as the eigenvector of P' associated with the unit eigenvalue. Starting from $u_0 = 0.20$, iterate $\pi_{t+1} = \pi_t P$ for 24 months, check the closed form of Problem 1 Q2, and plot u_t. Simulate the employment history of a single worker for 10^6 months and compare the fraction of time unemployed with $\bar{u}$. Which property of the chain does this illustrate?

2. Brock–Mirman. With $\alpha = 0.3$, $\beta = 0.96$, $\rho = 0.9$, $\sigma_\varepsilon = 0.01$, simulate 10^5 periods of the exact policy $K' = \alpha\beta A K^\alpha$ and of the linearised policy $K' = \bar{K} + \alpha(K - \bar{K}) + \bar{K}(A - 1)$ (Problem 7 Q3), with the same shocks. Compare the sample standard deviation of $\log K$ with the analytical value of Problem 6, and report the maximum percentage gap between the two capital paths.

3. Impulse responses. With the slides' calibration of Problem 7 Q4, compute g_K and g_A, and plot the responses of A, Y, C and K (in % of steady state) to a one-standard-deviation TFP shock (0.5%) over 200 quarters. Report the impact responses of Y and C and the peak dates of C and K. Explain the shapes.

4. Self-insurance. Solve the income-fluctuation problem of Problem 11 by value function iteration: u CRRA with $\sigma = 2$, $\beta = 0.95$, $r = 0.03$, $\underline{a} = 0$, income $y \in \{1 - \Delta, 1 + \Delta\}$ with persistence $\Pr(y' = y) = 0.9$, and an asset grid of 1,000 points on $[0, 20]$. For $\Delta = 0.3$, plot the saving policies $a' = g(a, y)$ with the 45-degree line and the CDF of the stationary distribution of assets. For $\Delta \in \{0.1, \ldots, 0.5\}$, report mean assets and the mass of consumers at the borrowing constraint. Comment.