2026-2027 – Econ 0107 – Macroeconomics I

Chapter 7 : Uncertainty

(Chapter 7 in Azzimonti, Krusell, McKay, Mukoyama and others's textbook)

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Outline

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2. Choice Under Uncertainty
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1. Stochastic Processes

Properties of Stochastic Processes

A stochastic process $\{X_t : t \in T\}$ is a collection of random variables indexed by time.

- **Conditional expectation**: $\mathbb{E}_t[X_{t+1}]$ conditions on all information available at t
- **Law of iterated expectations**: for $t < s < \tau$,

$$\mathbb{E}_t[\mathbb{E}_s[X_\tau]] = \mathbb{E}_t[X_\tau]$$

- **Covariance stationarity**: unconditional means and autocovariances do not depend on t
- **Ergodicity**: time averages converge to population moments, $(1/T)\sum_{t=1}^{T} X_t \to \mathbb{E}[X_t]$
- **Markov**: $\Pr[X_{t+k} = x \,|\, X_\tau, \forall \tau \leq t] = \Pr[X_{t+k} = x \,|\, X_t]$
- **Stationary distribution**: $\bar{\pi}(X)$ such that if $X_t \sim \bar{\pi}$, then $X_{t+1} \sim \bar{\pi}$

Covariance Stationary but not Ergodic

Example: draw $Z \sim \mathcal{N}(0, \sigma^2)$ once at $t = 0$, and set $X_t = Z$ for all t.

- **Covariance stationary**: $\mathbb{E}[X_t] = 0$, $\text{Var}(X_t) = \sigma^2$, and, since $X_t = X_{t-j} = Z$, $\gamma_j = \text{Cov}(X_t, X_{t-j}) = \text{Cov}(Z, Z) = \text{Var}(Z) = \sigma^2$ for all t and j (autocorrelation $\rho_j = 1$ at all lags)
- **Not ergodic**: the time average from one realisation is

$$\frac{1}{T}\sum_{t=1}^{T} X_t = Z \xrightarrow[T\to\infty]{} Z \neq 0 = \mathbb{E}[X_t]$$

  A long sample only reveals the particular draw of Z, never the population mean
- **Why**: autocovariances do not decay, $\gamma_j = \sigma^2$ for all $j \rightsquigarrow \sum_{j=0}^{\infty} |\gamma_j| < \infty$ (sufficient for ergodicity in the mean) fails

Markov Chains

- x_t takes values in $\mathcal{X} = \{\bar{x}^1, \ldots, \bar{x}^N\}$.
- Transition matrix P: $P_{ij} = \Pr[x_{t+1} = \bar{x}^j \mid x_t = \bar{x}^i]$.
- With $\pi_{t,i} = \Pr[x_t = \bar{x}^i]$, $\pi_{t+1} = \pi_t \times P$ reads

$$\begin{pmatrix} \pi_{t+1,1} & \cdots & \pi_{t+1,N} \end{pmatrix} = \begin{pmatrix} \pi_{t,1} & \cdots & \pi_{t,N} \end{pmatrix} \times \begin{pmatrix} P_{11} & \cdots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \cdots & P_{NN} \end{pmatrix}$$

- Each row sums to one. Given initial distribution π_0 (a $1 \times N$ vector):

$$\pi_{t+k} = \pi_t \times P^k$$

- **Stationary distribution**: $\bar{\pi} = \bar{\pi} \times P$, i.e., $\bar{\pi}'$ is the eigenvector of P' with unit eigenvalue.
- **Example**: employment/unemployment (slide 9)

Convergence of Markov Chains

- Not all Markov chains converge to a unique stationary distribution.
  - × $P = I$ (identity): any distribution is stationary (not unique)
  - × $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: unique stationary distribution $[1/2, 1/2]$ but any other distribution oscillates, never converges (see next slide)
- **Sufficient conditions for convergence and uniqueness**: State j is **reachable** from i if $\Pr(x_n = \bar{x}^j \mid x_0 = \bar{x}^i) > 0$ for some n. **Irreducible**: all states reachable from all others after some number of steps. **Aperiodic**: for each i, $\gcd\{n \geq 1 : \Pr(x_n = \bar{x}^i \mid x_0 = \bar{x}^i) > 0\} = 1$ (equivalently, $\exists N$ such that $\Pr(x_n = \bar{x}^i \mid x_0 = \bar{x}^i) > 0$ for all $n \geq N$).
- Note: gcd = greatest common divisor
- An irreducible chain has a unique $\bar{\pi}$. If also aperiodic, $\lim_{t\to\infty} \pi_0 P^t = \bar{\pi}$ for all π_0.
- If every entry of P is strictly positive, the chain has a unique stationary distribution and $\pi_0 P^t \to \bar{\pi}$ for every starting π_0.

A Periodic Chain

- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: unique stationary distribution $[1/2, 1/2]$ but starting with another distribution $\rightsquigarrow$ oscillations, no convergeence
- The chain switches state every period with certainty
- Unique $\bar{\pi}$: $[\bar{\pi}_1, \bar{\pi}_2]P = [\bar{\pi}_2, \bar{\pi}_1] = [\bar{\pi}_1, \bar{\pi}_2] \Rightarrow \bar{\pi}_1 = \bar{\pi}_2 = 1/2$
- No convergence: from $\pi_0 = [1, 0]$, $\pi_1 = [0, 1]$, $\pi_2 = [1, 0]$, $\ldots$; P^k alternates between I and P
- Eigenvalues 1 and -1: $\pi_k = \bar{\pi} + c\,(-1)^k v$ (v: eigenvector for -1), the second term never dies out
- Time averages still converge: a single path spends half its time in each state

Markov Chains: Employment/Unemployment Example

- States $\{e, u\}$, separation rate $0 < s < 1$, job-finding rate $0 < f < 1$, $e_t + u_t = 1$;

$$P = \begin{pmatrix} 1-s & s \\ f & 1-f \end{pmatrix}$$

- Law of motion of the unemployment rate: $u_{t+1} = (1-f)\,u_t + s\,(1-u_t)$ [AR(1)]
- Steady state: inflows = outflows, $s(1-\bar{u}) = f\,\bar{u} \rightsquigarrow \bar{u} = s/(s+f)$
- Convergence from any u_0: $u_{t+1} - \bar{u} = (1-s-f)(u_t - \bar{u})$, geometric since $|1-s-f| < 1$ ($1-s-f$ is the second eigenvalue of P)
- Monthly $s = 0.02$, $f = 0.30 \rightsquigarrow \bar{u} = 6.25\%$; gap to $\bar{u}$ halves in ≈ 1.8 months
- Ergodicity: the fraction of time a single worker spends unemployed $\to \bar{u}$

Note: Half-life of an AR(1)

$$x_{t+1} = \rho x_t \qquad \text{with } 0 < \rho < 1$$

- Iterating gives $x_t = \rho^t x_0$.
- The half-life h is the time it takes for x to fall to half its initial value:

$$\rho^h x_0 = \tfrac{1}{2} x_0 \implies h \ln \rho = -\ln 2 \implies h = -\frac{\ln 2}{\ln \rho} = \frac{\ln 2}{\ln(1/\rho)}.$$

- Since $0 < \rho < 1$, $\ln \rho < 0$, so $h > 0$.
- Approximation when ρ is close to 1: $\ln \rho \approx -(1 - \rho)$, so

$$h \approx \frac{0.69}{1 - \rho}.$$

AR(1) Process

$$x_t = \rho x_{t-1} + b\varepsilon_t + (1-\rho)\mu$$

with $\mathbb{E}_{t-1}[\varepsilon_t] = 0$, $\mathbb{E}_{t-1}[\varepsilon_t^2] = 1$, $\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_{t+s}] = 0$ for $s > 0$.

- If $|\rho| < 1$: stationary. Moving-average representation: $x_t = \mu + b\sum_{s=0}^{\infty} \rho^s \varepsilon_{t-s}$.
- **Moments**:
  - $\mathbb{E}[x_t] = \mu$
  - $\text{Var}(x_t) = b^2/(1-\rho^2)$
  - $\text{Corr}(x_t, x_{t+j}) = \rho^j$
- μ controls the level, b the volatility, ρ the persistence.

Vector Linear Stochastic Difference Equations

- $x_t \in \mathbb{R}^n$, $\varepsilon_t \in \mathbb{R}^m$ with $\mathbb{E}_t[\varepsilon_{t+1}] = 0$, $\mathbb{E}_t[\varepsilon_{t+1}\varepsilon'_{t+1}] = I$:

$$x_t = Ax_{t-1} + B\varepsilon_t + C$$

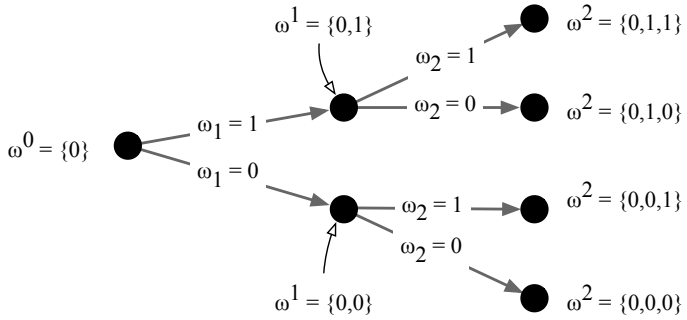
  - × Stationary if all eigenvalues of A are < 1 in absolute value
  - × Unconditional mean: $\mu = (I - A)^{-1}C$
  - × Result: If $\text{Cov}(y) = \Omega$, $\text{Cov}(My) = M\Omega M'$ (because $\Omega = \mathbb{E}[yy']$)
  - × Unconditional covariance: $\Gamma(0) = A\Gamma(0)A' + BB'$ (Lyapunov equation)
- **Impulse response function**: Effect of shock ε at t on x_{t+h} (no further shocks):

$$x_{t+h} - \mu = A^{h+1}(x_{t-1} - \mu) + A^h B\varepsilon_t$$

- $F(h) = A^h B\varepsilon$ is the IRF of x to ε. (the effect of an increase of 1 in ε staring from $x_{t-1} = \mu$)

2. Choice Under Uncertainty

Stochastic Events and Event Trees



- $\omega_t \in \Omega_t$: stochastic event at date t
- $\omega^t = \{\omega_0, \omega_1, \ldots, \omega_t\} \in \Omega^t$: history of events up to date t
- $\pi_t(\omega^t)$: date-0 probability of history ω^t
- $\pi_t(\omega^t|\omega^\tau)$: conditional probability of ω^t given ω^τ $(t > \tau)$
- Outcomes at t are functions of the full history: e.g., $a_t(\omega^t)$

Expected Utility and Risk Aversion

- Consumption process $\{c_t(\omega^t)\}$ ranked by:

$$\sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \pi_t(\omega^t) \beta^t u(c_t(\omega^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- **Risk aversion**: u concave $\Rightarrow$ less risky stream preferred (JENSEN's inequality)
- **Absolute risk aversion**: $-u''(c)/u'(c)$. CARA: $u(c) = -\exp(-\alpha c)$, coefficient $= \alpha$
- **Relative risk aversion**: $-c\,u''(c)/u'(c)$. CRRA: $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, coefficient $= \sigma$
- With CRRA, the EIS $= 1/\sigma$: a single parameter σ governs both risk aversion and intertemporal substitution (not with EPSTEIN-ZIN [1989] preferences)

Portfolio Choice: CRRA vs. CARA

- Investor allocates W between risk-free (R_f) and risky (Z) assets. Let $A =$ risky investment

$$\max_A \mathbb{E}_0 \left[u\left(R^f (W - A) + ZA \right) \right]$$

- **CRRA** ($u(c) = c^{1-\sigma}/(1-\sigma)$): FOC becomes

$$\mathbb{E}_0 \left[\left(R_f \left(1 - \frac{A}{W} \right) + Z \frac{A}{W} \right)^{-\sigma} (Z - R_f) \right] = 0$$

  Solution for A/W does not depend on W: rich and poor choose the **same risky share**

- **CARA** ($u(c) = -\exp(-\alpha c)$): FOC becomes

$$\mathbb{E}_0 \left[\exp\left(-\alpha (Z - R_f) A \right) (Z - R_f) \right] = 0$$

  Solution for A does not depend on W: rich have a **lower risky share**

- Data: rich hold higher risky share $\Rightarrow$ neither model fully matches, but CRRA is closer

3. The Stochastic Growth Model

Two-Period Economy: Setup

- Period 0: TFP known. Period 1: TFP $A_1(\omega_1)$ is random with probability $\pi_1(\omega_1)$
- **Preferences**:

$$U = u(C_0) + \beta \sum_{\omega_1 \in \Omega_1} \pi_1(\omega_1) u(C_1(\omega_1))$$

- **Resource constraints**:

$$K_1 + C_0 = K_0^\alpha + (1 - \delta) K_0$$

$$C_1(\omega_1) = A_1(\omega_1) K_1^\alpha + (1 - \delta) K_1 \qquad \forall \omega_1$$

- One resource constraint per state of the world in period 1
- The planner chooses C_0, K_1, and $\{C_1(\omega_1) : \forall \omega_1\}$ – a **contingent plan**

Two-Period Economy: Euler Equation

- Lagrangian with multipliers λ_0 and $\lambda_1(\omega_1)$.
- FOCs:

$$\begin{array}{lcll} \text{wrt } C_0 & : & u'(C_0) & = \lambda_0 \\ \text{wrt } K_1 & : & \lambda_0 & = \sum_{\omega_1 \in \Omega_1} \lambda_1(\omega_1)(\alpha A_1(\omega_1) K_1^{\alpha-1} + 1 - \delta) \\ \text{wrt } C_1(\omega_1) & : & \beta \pi_1(\omega_1) u'(C_1(\omega_1)) & = \lambda_1(\omega_1) \quad \forall \omega_1 \end{array}$$

- Combining:

$$u'(C_0) = \beta \sum_{\omega_1 \in \Omega_1} \pi_1(\omega_1) u'(C_1(\omega_1))(\alpha A_1(\omega_1) K_1^{\alpha-1} + 1 - \delta)$$

- **Stochastic Euler equation**: marginal utility loss from saving = expected marginal utility gain, weighted by state-contingent returns

Infinite-Horizon Economy: Preferences and Constraints

- **Preferences**:

$$U = \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \pi_t(\omega^t) \beta^t u(C_t(\omega^t))$$

- **Production**: $Y_t(\omega^t) = A_t(\omega^t) F(K_t(\omega^{t-1}), L_t(\omega^t))$
- Note: K_t depends on ω^{t-1} (chosen before ω_t realized)
- **Resource constraint**:

$$K_{t+1}(\omega^t) + C_t(\omega^t) = f(A_t(\omega^t), K_t(\omega^{t-1}))$$

where $f(A, K) \equiv AF(K, 1) + (1 - \delta)K$, for all t and ω^t

Infinite-Horizon Euler Equation

- FOC for $K_{t+1}(\omega^t)$ (affects production at $t+1$ for all ω^{t+1} extending ω^t):

$$\lambda_t(\omega^t) = \sum_{\{\omega^{t+1}|\omega^t\}} \lambda_{t+1}(\omega^{t+1}) f_2(A_{t+1}(\omega^{t+1}), K_{t+1}(\omega^t))$$

- FOC for $C_t(\omega^t)$: $\pi_t(\omega^t)\beta^t u'(C_t(\omega^t)) = \lambda_t(\omega^t)$
- **Euler equation**:

$$u'(C_t(\omega^t)) = \beta \sum_{\{\omega^{t+1}|\omega^t\}} \pi_{t+1}(\omega^{t+1}|\omega^t) u'(C_{t+1}(\omega^{t+1})) f_2(A_{t+1}(\omega^{t+1}), K_{t+1}(\omega^t))$$

- **Compact notation**:

$$u'(C_t) = \beta \mathbb{E}_t[u'(C_{t+1}) f_2(A_{t+1}, K_{t+1})]$$

Recursive Formulation

- Assume A follows a first-order Markov process with $\pi(A'|A)$
- **Bellman equation**:

$$V(A,K) = \max_{C,K' \geq 0} \left\{ u(C) + \beta \sum_{A'} \pi(A'|A) V(A',K') \right\}$$

subject to $K' + C = f(A,K)$

- **FOC**: $u'(C) = \beta \sum_{A'} \pi(A'|A) V_2(A',K')$
- **Envelope**: $V_2(A,K) = u'(C) f_2(A,K)$
- **Functional Euler equation** (with $K' = g(A,K)$):

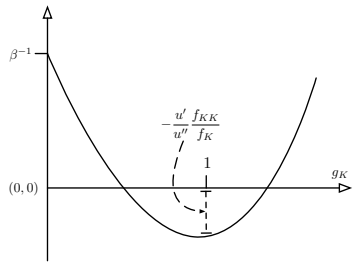
$$u'(f(A,K) - g(A,K)) = \beta \sum_{A'} \pi(A'|A) u'(f(A',g(A,K)) - g(A',g(A,K))) f_2(A',g(A,K))$$

Must hold for all (A,K)

Solving via Linearization

- Assume $A' = \rho A + (1-\rho)\bar{A} + \varepsilon'$, $\mathbb{E}[\varepsilon'] = 0$.
- Linearize around deterministic steady state $(\bar{A}, \bar{K})$.
- Let $\hat{X}_t \equiv X_t - \bar{X}$
- Let us look for a policy rule: $\hat{K}' = g_K \hat{K} + g_A \hat{A}$
- Linearize the functional Euler equation and matching coefficients on $\hat{K}$ yields:

$$g_K^2 - \left(1 + \beta^{-1} + \frac{u'}{u''}\frac{f_{KK}}{f_K}\right) g_K + \beta^{-1} = 0$$



Linearized Solution

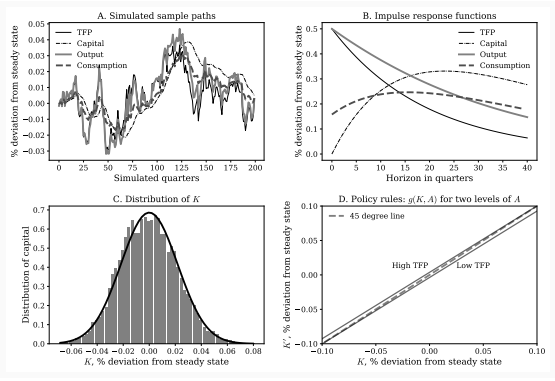
- The linearized policy rule is:

$$K' = \bar{K} + g_K(K - \bar{K}) + g_A(A - \bar{A})$$

- g_A can be solved similarly to g_K, using the linearized functional Euler equation
- Together with $A' = \bar{A} + \rho(A - \bar{A}) + \varepsilon'$: a system of two linear stochastic difference equations, analyzed using the tools of Section 7.1
- **Certainty equivalence**: After linearization, only $\mathbb{E}[\varepsilon']$ matters, not $\text{Var}[\varepsilon']$. It is as if there is no uncertainty and A' equals its conditional expectation

Numerical Illustration

Parameters: $f(A,K) = AK^{\alpha} + (1-\delta)K$, $\alpha = 0.3$, $\delta = 0.02$, $\rho = 0.95$, $\mathrm{SD}(\varepsilon) = 0.005$, $\bar{A} = 1$, $u = \log$, $\beta = 0.99$.



- TFP and Y: high-frequency variation. K and C: smooth
- On impact: Y jumps with A (K predetermined). C increases less than Y (saving)
- Policy rules shift with A; K fluctuates between their intersections with the 45-degree line

4. Competitive Equilibrium Under Uncertainty

Complete vs. Incomplete Markets

- **Complete markets**: For each (t, ω^t), a contract delivers one unit of good at that date-history and zero otherwise (**Arrow securities**). Any risk can be insured
- **Incomplete markets**: Some contracts unavailable. Some risks cannot be insured
- Setup:
  - × Households $i \in I$ with expected utility $\sum_{t=0}^{\infty} \sum_{\omega^t} \pi_t(\omega^t) \beta^t u(c_{i,t}(\omega^t))$
  - × Stochastic endowments $y_{i,t}(\omega^t)$
  - × Arrow-Debreu prices $p_t(\omega^t)$

Definition 13 (AD CE Under Uncertainty)

- An AD CE is $\{c^*_{i,t}(\omega^t) : \forall t, \omega^t\}$ for each $i \in I$, and $\{p_t(\omega^t) : \forall t, \omega^t\}$ such that
  1. for each i, $\{c^*_{i,t}(\omega^t)\}$ solves

$$\max_{\{c_t(\omega^t)\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \pi_t(\omega^t) u(c_t(\omega^t))$$

     subject to

$$\sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} p_t(\omega^t) c_t(\omega^t) = \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} p_t(\omega^t) y_{i,t}(\omega^t)$$

  2. $\sum_{i \in I} c^*_{i,t}(\omega^t) = \sum_{i \in I} y_{i,t}(\omega^t)$ for all (t, ω^t)

Characterization: Insurance and Risk Sharing

- FOC: $\beta^t \pi_t(\omega^t) u'(c_{i,t}(\omega^t)) = \lambda_i p_t(\omega^t)$
- **Insurance**: If prices are actuarially fair ($p_t(\omega^t) = \bar{p}_t \pi_t(\omega^t)$), then $u'(c_{i,t}(\omega^t)) = u'(c_{i,t}((\omega^t)'))$ for all $\omega^t, (\omega^t)' \Rightarrow$ **full insurance**: $c_{i,t}$ independent of ω^t
- **When are prices fair?** $p_t(\omega^t)/(\beta^t \pi_t(\omega^t))$ is a decreasing function of aggregate endowment $Y_t(\omega^t) = \sum_i y_{i,t}(\omega^t) \Rightarrow$ fair prices $\Leftrightarrow$ **no aggregate risk**. With identical CRRA: $p_t(\omega^t) \propto \beta^t \pi_t(\omega^t) Y_t(\omega^t)^{-\sigma}$
- **Risk sharing**: Take the ratio of FOCs for households i and j:

$$\frac{u'(c_{i,t}(\omega^t))}{u'(c_{j,t}(\omega^t))} = \frac{\lambda_i}{\lambda_j}$$

- λ_i/λ_j is constant across time and states $\Rightarrow$ $c_{i,t}$ depends only on **aggregate** endowment, not on $y_{i,t}$. All idiosyncratic risk is insured away

Sequential Trading: Arrow Securities

- For each ω_{t+1}, an asset traded at t pays 1 unit at $t+1$ if ω_{t+1} occurs. Price: $q_t(\omega_{t+1}|\omega^t)$
- **Budget constraint**:

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega_{t+1}) \leq y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$$

- **nPg**: $a_{i,t}(\omega^t) \geq -\sum_{\tau=t}^{\infty} \sum_{\omega^\tau} \tilde{q}_\tau^t(\omega^\tau) y_{i,\tau}(\omega^\tau)$
  where $\tilde{q}_{\tau+1}^t(\omega^{\tau+1}) = q_\tau(\omega_{\tau+1}|\omega^\tau)\tilde{q}_\tau^t(\omega^\tau)$ with $\tilde{q}_t^t(\omega^t) = 1$

Definition 14 (Sequential CE Under Uncertainty)

- A sequential CE is $\{c^*_{i,t}(\omega^t)\}$, $\{a^*_{i,t+1}(\omega^t)\}$ for each i, and $\{q_t(\omega^t)\}$ such that
  1. for each i, $\{c^*_{i,t}(\omega^t)\}$ and $\{a^*_{i,t+1}(\omega^t)\}$ solve

$$\max_{\{c_t(\omega^t),a_{t+1}(\omega^t)\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \pi(\omega^t) u(c_t(\omega^t))$$

     subject to

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega_{t+1}) = y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$$

     and the nPg condition, with $a_{i,0} = 0$
  2. $\sum_i (c^*_{i,t}(\omega^t) - y_{i,t}(\omega^t)) = 0$ and $\sum_i a^*_{i,t+1}(\omega^t) = 0$ for all (t, ω^t)
- The sequential and AD equilibria deliver the same allocations (same consolidation argument as Chapter 5)

Spanning and Complete Markets

- S states, N assets. Payoff matrix D $(S \times N)$. Portfolio $\theta \in \mathbb{R}^N$. Payoff vector: $D\theta$
- **Market span**: $\mathcal{M} \equiv \{z \in \mathbb{R}^S : z = D\theta \text{ for some } \theta \in \mathbb{R}^N\}$
- Markets are **complete** if $\mathcal{M} = \mathbb{R}^S$, i.e., $\text{rank}(D) = S$
- If complete: $\exists$ portfolios $\{\theta_j^A\}_{j=1}^S$ such that $D\theta_j^A = e_j$ (Arrow securities)
- Real-world assets pay off in multiple states, but completeness only requires that Arrow securities can be **replicated** by portfolios

5. Competitive Equilibrium in the Stochastic Growth Model

Setup

- Representative household with k_0 units of capital, one unit of labor (inelastic)
- **Production**: $y_t(\omega^t) = A_t(\omega^t)F(k_t(\omega^{t-1}), 1)$
- **Factor prices**:

$$r_t(\omega^t) = A_t(\omega^t)F_1(k_t(\omega^{t-1}), 1), \qquad w_t(\omega^t) = A_t(\omega^t)F_2(k_t(\omega^{t-1}), 1)$$

- **Budget**:

$$c_t(\omega^t) + k_{t+1}(\omega^t) = (r_t(\omega^t) + 1 - \delta)k_t(\omega^{t-1}) + w_t(\omega^t)$$

- **Household Euler equation**:

$$u'(c_t(\omega^t)) = \mathbb{E}_t[\beta u'(c_{t+1}(\omega^{t+1}))(r_{t+1}(\omega^{t+1}) + 1 - \delta)]$$

Definition 15 (Sequential CE, Stochastic Growth Model)

- A sequential CE is $\{c_t^*(\omega^t)\}$, $\{k_{t+1}^*(\omega^t)\}$, $\{r_t(\omega^t), w_t(\omega^t)\}$ for all (t, ω^t) such that
  1. $\{c_t^*(\omega^t)\}$ and $\{k_{t+1}^*(\omega^t)\}$ solve

$$\max_{\{c_t(\omega^t), k_{t+1}(\omega^t)\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \pi(\omega^t) u(c_t(\omega^t))$$

     subject to $c_t(\omega^t) + k_{t+1}(\omega^t) = (r_t(\omega^t) + 1 - \delta) k_t(\omega^{t-1}) + w_t(\omega^t)$ and the nPg condition
  2. $r_t(\omega^t) = A_t(\omega^t) F_1(k_t^*(\omega^{t-1}), 1)$ and $w_t(\omega^t) = A_t(\omega^t) F_2(k_t^*(\omega^{t-1}), 1)$
  3. $k_{t+1}^*(\omega^t) + c_t^*(\omega^t) = (1 - \delta) k_t^*(\omega^{t-1}) + A_t(\omega^t) F(k_t^*(\omega^{t-1}), 1)$
- Condition 3 is redundant (WALRAS' Law with CRS)
- The CE allocation coincides with the planner's solution; welfare theorems apply with goods indexed by (t, ω^t)

6. Incomplete Markets

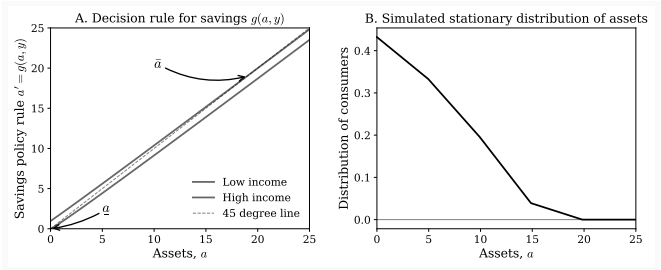
Incomplete-Market Consumption-Saving Problem

- Consumer with stochastic income y_t (Markov), single asset with gross return $1 + r$:

$$V(a, y) = \max_{a' \geq \underline{a}} \left\{ u(a + y - qa') + \beta \mathbb{E}[V(a', y')|y] \right\}$$

  where $q = 1/(1 + r)$
- Assume $\underline{a}$ is the natural debt limit (so that $a' \geq \underline{a}$ never binds)
- **Euler equation**: $u'(c) = (1 + r)\beta \mathbb{E}[u'(c')|y]$
- Single asset cannot insure against income risk $\Rightarrow$ consumption fluctuates with y_t
- **Partial self-insurance**: Accumulate savings to spend down during low-income periods. Fully smooth consumption is not optimal (would require consuming at the worst-case level forever)

Decision Rules and Stationary Distribution



- High income: accumulate savings up to $\bar{a}$ (upper line crosses 45-degree)
- Low income: spend down savings to $\underline{a}$ (borrowing constraint)
- Ergodic set: $[\underline{a}, \bar{a}]$; within it, assets drift up or down
- Stationary distribution: reflects accumulated luck/misfortune across consumers
- Contrast with complete markets: in the deterministic steady state of Chapter 5, $g(a) = a$ (45-degree line)

7. Summary

Key Takeaways (I)

1. **Stochastic processes**: Markov chains ($\pi_{t+k} = \pi_t P^k$), AR(1) (ρ persistence, b volatility), vector LSDE ($x_t = Ax_{t-1} + B\varepsilon_t + C$, IRF $= A^h B\varepsilon$)
2. **Expected utility**: $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$; risk aversion from concavity; CRRA ties together RRA and EIS via σ
3. **Portfolio choice**: CRRA $\Rightarrow$ constant risky share; CARA $\Rightarrow$ constant risky amount
4. **Stochastic Euler equation**: $u'(C_t) = \beta \mathbb{E}_t[u'(C_{t+1}) f_2(A_{t+1}, K_{t+1})]$; returns weighted by state-contingent marginal utilities

Key Takeaways (II)

5. **Linearization**: Quadratic in g_K (one stable root < 1); certainty equivalence (only $\mathbb{E}[\varepsilon']$ matters, not $\text{Var}[\varepsilon']$)
6. **Complete markets** (Def 13–14): Full risk sharing; idiosyncratic risk insured; $c_{i,t}$ depends only on aggregate endowment. Markets complete iff $\text{rank}(D) = S$.
7. **Stochastic CE of NGM** (Def 15): Same as planner's solution; welfare theorems hold with goods indexed by (t, ω^t)
8. **Incomplete markets**: Single asset, Euler inequality at borrowing constraint; partial self-insurance; stationary wealth distribution emerges from accumulated idiosyncratic shocks

